Homogeneous spaces
B. Komrakov seminar

FOUR–DIMENSIONAL PSEUDO–RIEMANNIAN HOMOGENEOUS SPACES. CLASSIFICATION OF COMPLEX PAIRS II

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Introduction

We consider classification of lower-dimensional homogeneous spaces an immediate continuation and global version of classification results obtained by Sophus Lie. Two-dimensional homogeneous spaces were classified locally by Sophus Lie [L1] and globally by G.D. Mostow [M]. (See also preprint [KTD], where the complete classification of two-dimensional homogeneous spaces, both locally and globally, is presented.) S. Lie also obtained some results in classification of three-dimensional homogeneous spaces and described all subalgebras in the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ (in terms of vector fields). A detailed account of these classifications can be found in [L2]. The local classification of all three-dimensional isotropically-faithful homogeneous spaces was obtained in [KT], and the classification (local and global) of all two- and three-dimensional pseudo-Riemannian isotropically-faithful homogeneous spaces was given in [DK].

The problem of classification of four-dimensional pseudo-Riemannian homogeneous spaces is interesting from the point of view of both geometry and physics, and not only in the case of signature $(1,3)$ (spaces of relativity theory) but also in the case of signature $(2,2)$ (twistors).

Let $(\overline{H},M)$ be a homogeneous space, $H=\overline{H}_x$ the stabilizer of an arbitrary point $x\in M$, and $(\bar{\mathfrak{h}},\mathfrak{h})$ the pair of Lie algebras corresponding to the pair $(\overline{H},H)$ of Lie groups.

Lemma. *Suppose that the homogeneous space $(\overline{H},M)$ admits an invariant pseudo-Riemannian metric. Then the isotropic representation of the pair $(\bar{\mathfrak{h}},\mathfrak{h})$*

$$\rho\colon \mathfrak{h}\to\mathfrak{gl}(\bar{\mathfrak{h}}/\mathfrak{h}),\qquad \rho(x)(\bar{x}+\mathfrak{h})=[x,\bar{x}]+\mathfrak{h}\quad (x\in\mathfrak{h},\ \bar{x}\in\bar{\mathfrak{h}})$$

is faithful. Moreover, there exists a basis of $\bar{\mathfrak{h}}/\mathfrak{h}$ such that $\rho(\mathfrak{h})$ lies in one of the following Lie algebras: $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$, or $\mathfrak{so}(2,2)$, which are the real forms of the complex Lie algebra $\mathfrak{so}(4,\mathbb{C})$.

In accordance with this, we divide solution of our problem into the following parts:

1. We find (up to conjugation) all possible forms the subalgebra $(\rho(\mathfrak{h}))^{\mathbb{C}}=\rho^{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}})$ can assume. This is equivalent to classifying (up to conjugation) subalgebras $\mathfrak{p}$ in the Lie algebra $\mathfrak{so}(4,\mathbb{C})$.
2. For each subalgebra $\mathfrak{p}$ obtained in (1), we find (up to equivalence of pairs) all complex pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ such that the subalgebra $\rho^{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}})$ is conjugate to $\mathfrak{p}$ (here $\mathfrak{g}=\mathfrak{h}^{\mathbb{C}}$).
3. For each complex pair $(\bar{\mathfrak{g}},\mathfrak{g})$, we find (up to equivalence of pairs) all its real forms $(\bar{\mathfrak{g}}^{\sigma},\mathfrak{g}^{\sigma})$, where σ is an anti-involution of the pair $(\bar{\mathfrak{g}},\mathfrak{g})$.
4. For each real pair obtained in (3), we construct all (up to isomorphism) corresponding homogeneous spaces.

This paper presents the complete proofs of the results of the first part (the classification of the complex pairs) of our work devoted to classification of four-dimensional homogeneous spaces with an invariant pseudo-Riemannian metric of arbitrary signature (for a summary of results see [K]). A similar classification for the case of Riemannian metric can be found in [I].

A detailed description of techniques we use for constructing pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ with a given faithful isotropic representation can be found in [KT].

CHAPTER I

ISOTROPICALLY–FAITHFUL PAIRS

1. Classification of subalgebras in the Lie algebra $\mathfrak{so}(4,\mathbb{C})$

Preliminaries:

1. For the sake of simplicity instead of the standard notation for a subalgebra of $\mathfrak{so}(4,\mathbb{C})$ such as

$$\mathfrak{g} = \left\{ \left. \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix} \right| x \in \mathbb{C} \right\},$$

where $|\lambda| < 1,\ -\frac{\pi}{2} < \arg\lambda \leqslant \frac{\pi}{2}$ or $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \frac{\pi}{2}$ we use the following notation:

$$\mathfrak{g} = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix}$$

$$|\lambda| < 1,\ -\frac{\pi}{2} < \arg\lambda \leqslant \frac{\pi}{2} \text{ or } |\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \frac{\pi}{2}.$$

Here we imply that variables denoted by Latin letters run through $\mathbb{C}$ and that parameters are denoted by small Greek letters.

2. To refer to subalgebras determined in Theorem 1 we use the following notation:

$$d.n,$$

where d is the dimension of the subalgebra; n is the number of the subalgebra in Theorem 1.

Theorem 1. *Any non-zero subalgebra of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ is conjugate (with respect to $GL(4,\mathbb{C})$) to one and only one of the following subalgebras:*

$\underline{\dim \mathfrak{g} = 1}$

1.1 $\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix}$ 1.2 $\begin{pmatrix} x & x & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -x & -x \end{pmatrix}$

$|\lambda| < 1,\ -\frac{\pi}{2} < \arg\lambda \leqslant \frac{\pi}{2}$ or $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \frac{\pi}{2}$

1.3 $\begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 1.4 $\begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\underline{dim\ \mathfrak{g} = 2}$

2.1 $\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}$ 2.2 $\begin{pmatrix} x & y & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}$

$|\lambda| < 1$ or $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi$

2.3 $\begin{pmatrix} x & y & 0 & x \\ 0 & -x & -x & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & x \end{pmatrix}$ 2.4 $\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2.5 $\begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix}$

$\underline{dim\ \mathfrak{g} = 3}$

3.1 $\begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix}$ 3.2 $\begin{pmatrix} x & y & 0 & z \\ 0 & \lambda x & -z & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}$

$\operatorname{Re}\lambda > 0$, or $\operatorname{Re}\lambda = 0,\ \operatorname{Im}\lambda \geqslant 0$

3.3 $\begin{pmatrix} 0 & y & 0 & z \\ 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{pmatrix}$ 3.4 $\begin{pmatrix} x & y & 0 & 0 \\ z & -x & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix}$

3.5 $\begin{pmatrix} 2x & y & 0 & 0 \\ 2z & 0 & -2y & 0 \\ 0 & -z & -2x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\underline{dim\ \mathfrak{g} = 4}$

4.1 $\begin{pmatrix} x & z & 0 & t \\ 0 & y & -t & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix}$ 4.2 $\begin{pmatrix} x & y & 0 & 0 \\ z & t & 0 & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix}$

4.3 $\begin{pmatrix} x & y & 0 & t \\ z & -x & -t & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & x \end{pmatrix}$

<u>$dim\ \mathfrak{g} = 5$</u>

$$5.1 \quad \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & 0 & -x & -z \\ 0 & 0 & -y & -t \end{pmatrix}$$

<u>$dim\ \mathfrak{g} = 6$</u>

$$6.1 \quad \begin{pmatrix} x & y & 0 & u \\ z & t & -u & 0 \\ 0 & v & -x & -z \\ -v & 0 & -y & -t \end{pmatrix}$$

Remark. *To simplify the computation, instead of $\mathfrak{so}(4,\mathbb{C})$ we use the linear Lie algebra 6.1, which is conjugate to $\mathfrak{so}(4,\mathbb{C})$.*

We divide the classification of all subalgebras of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ into two parts:
I. Classification of solvable subalgebras.
II. Classification of non-solvable subalgebras.

I. Classification of solvable subalgebras in $\mathfrak{so}(4,\mathbb{C})$.
Any maximal solvable subalgebra in $\mathfrak{so}(4,\mathbb{C})$ is conjugate to the following:

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} x & z & 0 & t \\ 0 & y & -t & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} \middle| \; x, y, z, t \in \mathbb{C} \right\}.$$

Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\bar{\mathfrak{g}}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

[,]	e_1	e_2	e_3	e_4
e_1	0	0	e_3	e_4
e_2	0	0	$-e_3$	e_4
e_3	$-e_3$	e_3	0	0
e_4	$-e_4$	$-e_4$	0	0 .

The group of automorphisms of $\bar{\mathfrak{g}}$ has the form:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -a & c & 0 \\ b & b & 0 & d \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ a & a & 0 & d \\ b & -b & c & 0 \end{pmatrix} \middle| \; d, f \in \mathbb{C}^*, \, b, c \in \mathbb{C} \right\}.$$

The composition series of $\bar{\mathfrak{g}}$ has the form:

$$\{0\} \subset \bar{\mathfrak{g}}_1 \subset \bar{\mathfrak{g}}_2 \subset \bar{\mathfrak{g}}, \text{ with } \bar{\mathfrak{g}}_1 = \mathbb{C}e_3 \oplus \mathbb{C}e_4, \; \bar{\mathfrak{g}}_2 = \mathbb{C}e_1 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4.$$

To classify all subalgebras in $\bar{\mathfrak{g}}$, we shall use the following algorithm (see [KT1]):

1) We describe (up to the group $\mathcal{A}$) all subalgebras $\mathfrak{g}_1$ of the ideal $\bar{\mathfrak{g}}_1$ and construct for each $\mathfrak{g}_1$:

a) $N(\mathfrak{g}_1) \cap \bar{\mathfrak{g}}_2$,

b) the subgroup $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}_1)$ of $\mathcal{A}$ that consists of all automorphisms of $\bar{\mathfrak{g}}$ preserving the subalgebra $\mathfrak{g}_1$.

2) We describe (up to the group $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}_1)$) all subalgebras $\mathfrak{g}_2$ of the Lie algebra $N(\mathfrak{g}_1) \cap \bar{\mathfrak{g}}_2$ such that $\mathfrak{g}_2 \cap \bar{\mathfrak{g}}_1 = \mathfrak{g}_1$ and construct for each $\mathfrak{g}_2$:

a) $N(\mathfrak{g}_2) \cap \bar{\mathfrak{g}}$,

b) the subgroup $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}_2)$ of $\mathcal{A}$ that consists of all automorphisms of $\bar{\mathfrak{g}}$ preserving the subalgebra $\mathfrak{g}_1$.

3) And so on.

1. Any subalgebra $\mathfrak{g}_1$ of the Lie algebra $\bar{\mathfrak{g}}_1 = \mathbb{C}e_3 \oplus \mathbb{C}e_4$ is conjugate to one and only one of the following subalgebras:

$$1) \; \{0\}, \quad 2) \; \mathbb{C}(e_3 + e_4), \quad 3) \; \mathbb{C}e_3, \quad 4) \; \mathbb{C}e_3 \oplus \mathbb{C}e_4.$$

$\mathfrak{g}_1$	$N(\mathfrak{g}_1) \cap \bar{\mathfrak{g}}_2$	$\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}_1)$
$\{0\}$	$\bar{\mathfrak{g}}_2$	$\mathcal{A}$
$\mathbb{C}(e_3 + e_4)$	$\bar{\mathfrak{g}}_2$	$\mathcal{A}(c = d)$
$\mathbb{C}e_3$	$\bar{\mathfrak{g}}_2$	$\mathcal{A}_1$
$\mathbb{C}e_3 \oplus \mathbb{C}e_4$	$\bar{\mathfrak{g}}_2$	$\mathcal{A}$

Here

$$\mathcal{A}_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -a & c & 0 \\ b & b & 0 & d \end{pmatrix} \middle| \; d, f \in \mathbb{C}^*, \, b, c \in \mathbb{C} \right\}.$$

2. Find subalgebras $\mathfrak{g}_2$ of $N(\mathfrak{g}_1) \cap \bar{\mathfrak{g}}_2$ such that $\mathfrak{g}_2 \cap \bar{\mathfrak{g}}_1 = \mathfrak{g}_1$.

$\mathfrak{g}_1$	$\mathfrak{g}_2$	$N(\mathfrak{g}_2)\cap\bar{\mathfrak{g}}$	$\mathrm{Aut}(\bar{\mathfrak{g}},\mathfrak{g}_2)$
$\{0\}$	$\{0\}$	$\bar{\mathfrak{g}}$	$\mathcal{A}$
	$\mathbb{C}e_1$	$\mathbb{C}e_1\oplus\mathbb{C}e_2$	$\mathcal{A}(a=b=0)$
$\mathbb{C}(e_3+e_4)$	$\mathbb{C}(e_3+e_4)$	$\mathbb{C}e_1\oplus\mathbb{C}e_3\oplus\mathbb{C}e_4$	$\mathcal{A}(c=d)$
	$\mathbb{C}e_1\oplus\mathbb{C}(e_3+e_4)$	$\mathbb{C}e_1\oplus\mathbb{C}(e_3+e_4)$	$\mathcal{A}(a=b,\ c=d)$
$\mathbb{C}e_3$	$\mathbb{C}e_3$	$\bar{\mathfrak{g}}$	$\mathcal{A}_1$
	$\mathbb{C}e_1\oplus\mathbb{C}e_3$	$\mathbb{C}e_1\oplus\mathbb{C}e_2\oplus\mathbb{C}e_3$	$\mathcal{A}_1(b=0)$
$\mathbb{C}e_3\oplus\mathbb{C}e_4$	$\mathbb{C}e_3\oplus\mathbb{C}e_4$	$\bar{\mathfrak{g}}$	$\mathcal{A}$
	$\mathbb{C}e_1\oplus\mathbb{C}e_3\oplus\mathbb{C}e_4$	$\bar{\mathfrak{g}}$	$\mathcal{A}$

3. Find subalgebras $\mathfrak{g}$ of $N(\mathfrak{g}_2)\cap\bar{\mathfrak{g}}$ such that $\mathfrak{g}\cap\bar{\mathfrak{g}}_2=\mathfrak{g}_2$.

$\mathfrak{g}_2$	$\mathfrak{g}$
$\{0\}$	$\{0\},\ \mathbb{C}(e_1+\alpha e_2+\beta e_3+\gamma e_4),\ \alpha\neq 0,\ \mathbb{C}e_2$
$\mathbb{C}e_1$	$\mathbb{C}e_1,\ \mathbb{C}e_1\oplus\mathbb{C}e_2$
$\mathbb{C}(e_3+e_4)$	$\mathbb{C}(e_3+e_4)$
$\mathbb{C}e_1\oplus\mathbb{C}(e_3+e_4)$	$\mathbb{C}e_1\oplus\mathbb{C}(e_3+e_4)$
$\mathbb{C}e_3$	$\mathbb{C}e_3,\ \mathbb{C}(e_1+\alpha e_2+\beta e_4)\oplus\mathbb{C}e_3,\ \alpha\neq 0,\ \mathbb{C}e_2\oplus\mathbb{C}e_3$
$\mathbb{C}e_1\oplus\mathbb{C}e_3$	$\mathbb{C}e_1\oplus\mathbb{C}e_3,\ \mathbb{C}e_1\oplus\mathbb{C}e_2\oplus\mathbb{C}e_3$
$\mathbb{C}e_3\oplus\mathbb{C}e_4$	$\mathbb{C}e_3\oplus\mathbb{C}e_4,\ \mathbb{C}(e_1+\alpha e_2)\oplus\mathbb{C}e_3\oplus\mathbb{C}e_4,$ $\mathrm{Re}\,\alpha>0$ or $\mathrm{Re}\,\alpha=0,\ \mathrm{Im}\,\alpha>0,\ \mathbb{C}e_2\oplus\mathbb{C}e_3\oplus\mathbb{C}e_4$
$\mathbb{C}e_1\oplus\mathbb{C}e_3\oplus\mathbb{C}e_4$	$\mathbb{C}e_1\oplus\mathbb{C}e_3\oplus\mathbb{C}e_4,\ \mathbb{C}e_1\oplus\mathbb{C}e_2\oplus\mathbb{C}e_3\oplus\mathbb{C}e_4$

Let $\mathfrak{g}=\mathbb{C}(e_1+\alpha e_2+\beta e_3+\gamma e_4),\ \alpha\neq 0,\ \pi\in\mathcal{A}_1$.

$\pi(e_1+\alpha e_2+\beta e_3+\gamma e_4)=e_1+ae_3+be_4+\alpha e_2-\alpha ae_3+\alpha be_4+\beta ce_3+\gamma de_4=e_1+\alpha e_2+((1-\alpha)a+\beta c)e_3+((1+\alpha)b+\gamma d)e_4$.

Consider the following cases:

1) $\alpha\notin\{-1,1\}$.

Then $\mathfrak{g}\cong\mathbb{C}(e_1+\alpha e_2)$.

Using $\pi_1\in\mathcal{A}\backslash\mathcal{A}_1$, we obtain $\mathrm{Re}\,\alpha>0$ or $\mathrm{Re}\,\alpha=0,\ \mathrm{Im}\,\alpha>0$.

2) $\alpha=1$.

Then $\mathfrak{g}\cong\mathbb{C}(e_1+e_2+e_3)$(if $\beta\neq 0$) or $\mathfrak{g}\cong\mathbb{C}(e_1+e_2)$(if $\beta=0$).

3) $\alpha=-1$.

Using $\pi_1\in\mathcal{A}\backslash\mathcal{A}_1$, we obtain that $\mathfrak{g}\cong\mathbb{C}(e_1+e_2+e_3)$(if $\gamma\neq 0$) or $\mathfrak{g}\cong\mathbb{C}(e_1+e_2)$(if $\gamma=0$).

Let $\mathfrak{g}=\mathbb{C}(e_1+\alpha e_2+\beta e_4)\oplus\mathbb{C}e_3,\ \alpha\neq 0,\ \pi\in\mathcal{A}_1(a=0)$.

$\pi(e_1+\alpha e_2+\beta e_4)=e_1+be_4+\alpha e_2+\alpha be_4+\beta de_4=e_1+\alpha e_2+((1+\alpha)b+\beta d)e_4$.

Consider the following cases:

1) $\alpha\neq -1$.

Then $\mathfrak{g}\cong\mathbb{C}(e_1+\alpha e_2)\oplus\mathbb{C}e_3$.

2) $\alpha=-1$.

Then $\mathfrak{g}\cong\mathbb{C}(e_1-e_2+e_4)\oplus\mathbb{C}e_3$(if $\beta\neq 0$) or $\mathfrak{g}\cong\mathbb{C}(e_1-e_2)\oplus\mathbb{C}e_3$(if $\beta=0$).

4. Thus, we have shown that every subalgebra of $\bar{\mathfrak{g}}$ is conjugate (up to the group $\mathcal{A}$) to one and only one of the following subalgebras:

0.1. $\{0\}$;
1.1. $\mathbb{C}(e_1 + \lambda e_2)$, $\lambda \neq 0$, $\operatorname{Re}\lambda > 0$ or $\operatorname{Re}\lambda = 0$, $\operatorname{Im}\lambda \geqslant 0$;
1.2. $\mathbb{C}(e_1 + e_2 + e_3)$;
1.3. $\mathbb{C}e_3$;
1.4. $\mathbb{C}(e_3 + e_4)$;
1.5. $\mathbb{C}e_2$;
2.1. $\mathbb{C}e_1 \oplus \mathbb{C}e_2$;
2.2. $\mathbb{C}(e_1 + \lambda e_2) \oplus \mathbb{C}e_3$;
2.3. $\mathbb{C}(e_1 - e_2 + e_4) \oplus \mathbb{C}e_3$;
2.4. $\mathbb{C}e_1 \oplus \mathbb{C}(e_3 + e_4)$;
2.5. $\mathbb{C}e_3 \oplus \mathbb{C}e_4$;
2.6. $\mathbb{C}e_2 \oplus \mathbb{C}e_3$;
3.1. $\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$;
3.2. $\mathbb{C}(e_1 + \lambda e_2) \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$, $\operatorname{Re}\lambda > 0$ or $\operatorname{Re}\lambda = 0$, $\operatorname{Im}\lambda \geqslant 0$;
3.3. $\mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$;
4.1. $\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$.

5. Consider subalgebras of $\bar{\mathfrak{g}}$ up to conjugation with respect to GL(4,$\mathbb{C}$).

$\underline{\dim\ \mathfrak{g} = 1}$

$$1.1 \quad \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -\lambda x \end{pmatrix} \qquad 1.2 \quad \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -x & -x \end{pmatrix}$$

$\operatorname{Re}\lambda > 0$ or $\operatorname{Re}\lambda = 0$, $\operatorname{Im}\lambda \geqslant 0$

$$1.3 \quad \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix} \qquad 1.4 \quad \begin{pmatrix} 0 & x & 0 & x \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix}$$

$$1.5 \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x \end{pmatrix}$$

The algebra 1.5 is conjugate to the algebra 1.1 ($\lambda = 0$) by means of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

In case 1.1 ($\lambda \neq 0$), the Lie algebras corresponding to λ and $\frac{1}{\lambda}$ are conjugate by means of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus we can assume that $|\lambda| < 1$ or $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi$. For other distinct values of λ the algebras of type 1.1 are not conjugate.

Now we show that other algebras are not conjugate to each other.

1.1($\lambda \notin \{0,1\}$): $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, eigenvalues: $x, -x, \lambda x, -\lambda x$.

1.1($\lambda = 1$): $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, eigenvalues: $x, -x$, each element of $\mathfrak{g}$ is semi-simple.

1.2: $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, eigenvalues: $x, -x$, each non-zero element of $\mathfrak{g}$ is not semi-simple.

1.1($\lambda = 0$): $\det X = 0$, for all $X \in \mathfrak{g}$, $\mathfrak{g}^3 \neq \{0\}$.

1.3: $\det X = 0$, for all $X \in \mathfrak{g}$, $\mathfrak{g}^2 = \{0\}$.

1.4: $\det X = 0$, for all $X \in \mathfrak{g}$, $\mathfrak{g}^2 \neq \{0\}$, $\mathfrak{g}^3 = \{0\}$.

$\underline{\dim\ \mathfrak{g} = 2}$

2.1 $\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}$ 2.2 $\begin{pmatrix} x & y & 0 & 0 \\ 0 & \lambda x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}$

2.3 $\begin{pmatrix} x & y & 0 & x \\ 0 & -x & -x & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & x \end{pmatrix}$ 2.4 $\begin{pmatrix} x & y & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & 0 \end{pmatrix}$

2.5 $\begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \end{pmatrix}$ 2.6 $\begin{pmatrix} 0 & y & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{pmatrix}$

The algebra 2.6 is conjugate to the algebra 2.2 ($\lambda = 0$) by means of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In case 2.2 ($\lambda \neq 0$), the Lie algebras corresponding to λ and $\frac{1}{\lambda}$ are conjugate by means of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we can assume that $|\lambda| < 1$ or $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi$. For other distinct values of λ the algebras of type 2.2 are not conjugate.

Now we show that other algebras are not conjugate to each other.

2.1: $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 0$, eigenvalues: $x, -x$.

2.2($\lambda \notin \{0,1\}$): $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 1$, eigenvalues: $x, -x, \lambda x, -\lambda x$.

2.2($\lambda = 1$): $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 0$, eigenvalues: $x, -x$.

2.3: $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 1$, eigenvalues: $x, -x$.

2.2($\lambda = 0$): $\det X = 0$, for all $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 1$, $\mathfrak{n}^2\mathfrak{g} = \{0\}$.

2.4: $\det X = 0$, for all $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 1$, $\mathfrak{n}^2\mathfrak{g} \neq \{0\}$.

2.5: $\det X = 0$, for all $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 0$.

$$\underline{\dim\ \mathfrak{g} = 3}$$

$$3.1 \quad \begin{pmatrix} x & z & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -z & -y \end{pmatrix} \qquad 3.2 \quad \begin{pmatrix} x & y & 0 & z \\ 0 & \lambda x & -z & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & -y & -\lambda x \end{pmatrix}$$

$$\operatorname{Re}\lambda > 0 \text{ or } \operatorname{Re}\lambda = 0,\ \operatorname{Im}\lambda \geqslant 0$$

$$3.3 \quad \begin{pmatrix} 0 & y & 0 & z \\ 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -y & -x \end{pmatrix}$$

In case 3.2, the Lie algebras corresponding to distinct values of λ are not conjugate.

Show that algebras 3.1, 3.2 and 3.3 are not conjugate to each other.

3.1: $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 1$, eigenvalues: $x, y, -x, -y$.

3.2($\lambda = 1$): $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 1$, eigenvalues: $x, -x$.

3.2($\lambda \notin \{0,1\}$): $\det X \neq 0$, for all non-zero $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 2$.

3.2($\lambda = 0$): $\det X = 0$, for all $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 2$, there exist no non-zero $v \in \mathbb{C}^4$ such that $\mathfrak{g}(v) = 0$.

3.3: $\det X = 0$, for all $X \in \mathfrak{g}$, $\dim \mathcal{D}\mathfrak{g} = 2$, there exist a non-zero $v \in \mathbb{C}^4$ such that $\mathfrak{g}(v) = 0$ (for example, $v = (0,0,1,0)$).

Remark. *Let*

$$\mathfrak{g}_1 = \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{g}_2 = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{g}_3 = \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The algebra 1.3 is conjugate to the algebra $\mathfrak{g}_1$ by means of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}.$$

The algebra 1.4 is conjugate to the algebra $\mathfrak{g}_2$ by means of the matrix B.
The algebra 2.4 is conjugate to the algebra $\mathfrak{g}_3$ by means of the matrix B.

We have proved the following

Proposition 1. *Any non-zero solvable subalgebra of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ is conjugate to one and only one of the following subalgebras: 1.1, 1.2, 1.3, 1.4, 2.1, 2.2, 2.3, 2.4, 2.5, 3.1, 3.2, 3.3, 4.1.*

II. Classification of non-solvable subalgebras in $\mathfrak{so}(4,\mathbb{C})$.

Let $\mathfrak{g}$ be a non-solvable subalgebra of $\mathfrak{so}(4,\mathbb{C})$. Then $\mathfrak{g}$ contains a non-trivial semisimple Levi subalgebra $\mathfrak{a}$.

Any semisimple subalgebra of $\mathfrak{so}(4,\mathbb{C})$ is conjugate to one and only one of the following subalgebras: 3.4, 3.5, 6.1.

Subalgebra 3.5 is maximal in $\mathfrak{so}(4,\mathbb{C})$; subalgebra 6.1 is conjugate to $\mathfrak{so}(4,\mathbb{C})$.

In the case 3.4 classification of non-solvable subalgebras is equivalent to classification of subalgebras in $\mathfrak{sl}(2,\mathbb{C})$. It can be easily obtained, and we find out that any $\mathfrak{g}$ is conjugate to one and only one of the following subalgebras: 4.2, 4.3, 5.1.

We have proved the following

Proposition 2. *Any non-solvable subalgebra of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ is conjugate to one and only one of the following subalgebras: 3.4, 3.5, 4.2, 4.3, 5.1, 6.1.*

The results of the Theorem 1 are immediate from Propositions 1 and 2.

2. Method of classification of pairs

A detailed description of techniques we use for constructing pairs with a given faithful isotropic representation can be found in [KT]. Recall briefly some basic definitions and results from [KT].

A *generalized module* is a pair $(\mathfrak{g}, U)$, where $\mathfrak{g}$ is a Lie algebra and U is a $\mathfrak{g}$-module. A generalized module $(\mathfrak{g}, U)$ is said to be *faithful* if the $\mathfrak{g}$-module U is faithful. The *dimension of a generalized module* $(\mathfrak{g}, U)$ is the dimension of the vector space U.

Assume that V is a vector space and $\mathfrak{g}$ is a subspace of V. The pair $(V, \mathfrak{g})$ supplied with a bilinear mapping $B : \mathfrak{g} \times V \to V$, $(x, v) \mapsto x.v$ is called a *virtual pair* if the following conditions are satisfied:

(1) $\mathfrak{g}.\mathfrak{g} \subset \mathfrak{g}$;

(2) the restriction of B to $\mathfrak{g} \times \mathfrak{g}$ provides $\mathfrak{g}$ with the structure of a Lie algebra ($[x, y] = x.y$);
(3) V is a $\mathfrak{g}$-module with respect to B.

To any virtual pair $(V, \mathfrak{g})$ we can naturally assign the generalized module $(\mathfrak{g}, V/\mathfrak{g})$, which is said to be *associated with the virtual pair* $(V, \mathfrak{g})$.

The *isotropic representation of a virtual pair* $(V, \mathfrak{g})$ is the mapping

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V/\mathfrak{g})$$

defined by

$$\rho(x)(v + \mathfrak{g}) = x.v + \mathfrak{g} \quad \text{for all } v \in V, x \in \mathfrak{g}.$$

The virtual pair $(V, \mathfrak{g})$ is called *isotropically-faithful* if the homomorphism ρ is injective. It is obvious that a virtual pair $(V, \mathfrak{g})$ is isotropically-faithful if and only if the associated generalized module $(\mathfrak{g}, V/\mathfrak{g})$ is faithful.

Suppose $\bar{\mathfrak{g}}$ is a finite-dimensional Lie algebra and $\mathfrak{g}$ is a subalgebra of $\bar{\mathfrak{g}}$. Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ can be regarded as a virtual pair with respect to ordinary commutation restricted to $\mathfrak{g} \times \bar{\mathfrak{g}}$. The *isotropic representation of a pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the isotropic representation of the corresponding virtual pair. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is called *isotropically-faithful* if its isotropic representation is an injection.

Two pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are said to be *equivalent* if there exists an isomorphism of Lie algebras $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_2$ such that $\pi(\mathfrak{g}_1) = \mathfrak{g}_2$.

Definition. Suppose $(\mathfrak{g}, U)$ is a generalized module and $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is a linear mapping such that

$$q([x, y]) = x.q(y) - y.q(x) \quad \text{for all } x, y \in \mathfrak{g}. \tag{1}$$

Then the mapping q is called a *virtual structure* on the generalized module $(\mathfrak{g}, U)$.

Proposition 1. *Suppose q is a virtual structure on a generalized module $(\mathfrak{g}, U)$. Put $V_q = \mathfrak{g} \times U$. Then the bilinear mapping $\mathfrak{g} \times V_q \to V_q$ given by*

$$x.(y, u) = ([x, y] + q(x)(u), x.u) \quad \textit{for all } x, y \in \mathfrak{g}, u \in U \tag{2}$$

defines the virtual pair $(V_q, \mathfrak{g})$.

So, to any virtual structure on a generalized module $(\mathfrak{g}, U)$ we assign the virtual pair $(\mathfrak{g} \times U, \mathfrak{g})$ defined by formula (2). Moreover, any virtual pair $(V, \mathfrak{g})$ with the associated generalized module $(\mathfrak{g}, U)$ can be constructed in this way.

Suppose q_1 and q_2 are virtual structures on a generalized module $(\mathfrak{g}, U)$. We say that q_1 and q_2 are *equivalent* if the virtual pairs $(V_{q_1}, \mathfrak{g})$ and $(V_{q_2}, \mathfrak{g})$ are isomorphic, i.e. if there exists an isomorphism of vector spaces $H : V_{q_1} \to V_{q_2}$ such that the following conditions hold:

(*a*) $H(\mathfrak{g}) = \mathfrak{g}$;
(*b*) $H(x.v) = H(x).H(v) \quad \text{for all } x \in \mathfrak{g}, v \in V_{q_1}$.

Proposition 2. *Suppose q_1 and q_2 are virtual structures on a generalized module $(\mathfrak{g}, U)$ and there exists a mapping $h \in \mathcal{L}(U, \mathfrak{g})$ such that $q_1(x) - q_2(x) = x.h$ for all $x \in \mathfrak{g}$. Then the virtual structures q_1 and q_2 are equivalent.*

Thus, classification (up to isomorphism) of all virtual pairs $(V, \mathfrak{g})$ for a given generalized module $(\mathfrak{g}, U)$ reduces to classification of all virtual structures on the generalized module $(\mathfrak{g}, U)$ (up to equivalence).

Let $(\mathfrak{g}, U)$ be a faithful four-dimensional generalized module over the field $\mathbb{C}$. Suppose $\mathcal{E} = \{e_1, \dots, e_n\}$ is a basis of the Lie algebra $\mathfrak{g}$ ($n = \dim \mathfrak{g}$) and $\mathcal{U} = \{u_1, u_2, u_3, u_4\}$ is a basis of the vector space U.

For $x \in \mathfrak{g}$, by $A(x)$ and $B(x)$ denote the matrices of the mappings

$$\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g} \quad \text{and} \quad x_U : U \to U$$

in the bases $\mathcal{E}$ and $\mathcal{U}$ respectively. Then $A(x) \in \mathrm{Mat}_{n\times n}(\mathbb{C}), B(x) \in \mathrm{Mat}_{4\times 4}(\mathbb{C})$, and the mapping

$$\rho : \mathfrak{g} \to \mathfrak{so}(4, \mathbb{C}), \ x \mapsto B(x)$$

is an injection. This allows to identify the Lie algebra $\mathfrak{g}$ with a certain subalgebra of the Lie algebra $\mathfrak{so}(4, \mathbb{C})$. Without loss of generality it can be assumed that $\mathfrak{g}$ is one of the subalgebras of $\mathfrak{so}(4, \mathbb{C})$ determined in Theorem 1.

We can identify the set of mappings $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ with the set of mappings $C : \mathfrak{g} \to \mathrm{Mat}_{n\times 4}(\mathbb{C})$, where $C(x)$ is the matrix of the mapping $q(x)$ with respect to the bases fixed before.

In the sequel the mapping $C : \mathfrak{g} \to \mathrm{Mat}_{n\times 4}(\mathbb{C})$ will be called a virtual structure on the generalized module $(\mathfrak{g}, U)$ if the corresponding mapping q is a virtual structure.

Proposition 3. *A necessary and sufficient condition for a mapping*

$$C : \mathfrak{g} \to \mathrm{Mat}_{n\times 4}(\mathbb{C})$$

to be a virtual structure on the generalized module $(\mathfrak{g}, U)$ is that the following condition hold:

$$C([x, y]) = A(x)C(y) - C(y)B(x) - A(y)C(x) + C(x)B(y) \quad \text{for } x, y \in \mathfrak{g}. \tag{3}$$

Proposition 4. *Suppose C_1 and C_2 are virtual structures on the generalized module $(\mathfrak{g}, U)$ and there exists a matrix $H \in \mathrm{Mat}_{n\times 4}(\mathbb{C})$ such that for all $x \in \mathfrak{g}$ the following condition holds:*

$$C_1(x) - C_2(x) = A(x)H - HB(x). \tag{4}$$

Then C_1 and C_2 are equivalent.

Remark. *Note that all expressions in* (3) *and* (4) *are linear in $x, y \in \mathfrak{g}$. Therefore, in order to ensure that these conditions are satisfied for all $x, y \in \mathfrak{g}$, we must only check that they hold for $x, y \in \mathcal{E} = \{e_1, \dots, e_n\}$.*

Suppose $(V, \mathfrak{g})$ is a virtual pair and $(\mathfrak{g}, U)$, where $U = V/\mathfrak{g}$, is the generalized module associated with $(V, \mathfrak{g})$.

Proposition 5. *Let $\mathfrak{h}$ be a nilpotent subalgebra of $\mathfrak{g}$.*

(1) *A necessary and sufficient condition for the $\mathfrak{h}$-module V to be a direct sum of primary components is that the $\mathfrak{h}$-modules $\mathfrak{g}$ and U be direct sums of primary components.*
(2) *There exists a section $s : U \to V$ of the canonical surjection $\pi : V \to U$ such that for every $\alpha \in \mathfrak{h}^*$ the following condition holds:*

$$s(U^\alpha(\mathfrak{h})) \subset V^\alpha(\mathfrak{h}). \tag{5}$$

Suppose s is a section of the canonical surjection $\pi : V \to U$. We say that s is *consistent with the subalgebra* $\mathfrak{h}$ if

$$s(U^\alpha(\mathfrak{h})) \subset V^\alpha(\mathfrak{h}) \quad \text{for all } \alpha \in \mathfrak{h}^*.$$

From Proposition 5(2) it follows that there always exists such a section.

Proposition 6. *Suppose s is a section of the canonical surjection $\pi : V \to U$ consistent with the subalgebra $\mathfrak{h}$. Then the corresponding virtual structure $q_s : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ on the generalized module $(\mathfrak{g}, U)$ satisfies the following condition:*

$$q_s(\mathfrak{g}^\alpha(\mathfrak{h}))(U^\beta(\mathfrak{h})) \subset \mathfrak{g}^{\alpha+\beta}(\mathfrak{h}) \quad \textit{for } \alpha, \beta \in \mathfrak{h}^*. \tag{6}$$

We say that a virtual structure q on $(\mathfrak{g}, U)$ is *primary* (with respect to $\mathfrak{h}$) if q satisfies condition (6). From Propositions 5(2) and 6 it follows that every virtual structure is equivalent to a certain primary virtual structure.

Proposition 7. *Suppose q is a primary (with respect to $\mathfrak{h}$) virtual structure on the generalized module $(\mathfrak{g}, U)$ and $(V_q, \mathfrak{g})$ is the corresponding virtual pair. Then*

$$V_q^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \quad \textit{for all } \alpha \in \mathfrak{h}^*.$$

A virtual pair $(V, \mathfrak{g})$ is said to be *trivial* if there exists a submodule U of the $\mathfrak{g}$-module V such that $V = U \oplus \mathfrak{g}$. Note that a trivial virtual pair $(V, \mathfrak{g})$ is uniquely defined (up to isomorphism) by the corresponding generalized module $(\mathfrak{g}, V/\mathfrak{g})$.

Proposition 8. *Let q be a virtual structure on the generalized module $(\mathfrak{g}, U)$. Then a necessary and sufficient condition for the virtual pair $(V_q, \mathfrak{g})$ to be trivial is that q be equivalent to the zero mapping of $\mathfrak{g}$ into $\mathcal{L}(U, \mathfrak{g})$.*

Proposition 9. *If $\mathfrak{g}$ is a semisimple Lie algebra, then every virtual pair $(V, \mathfrak{g})$ is trivial.*

A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is called *trivial* if there exists a commutative ideal $\mathfrak{a}$ in the Lie algebra $\bar{\mathfrak{g}}$ such that $\mathfrak{g} \oplus \mathfrak{a} = \bar{\mathfrak{g}}$. Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a trivial pair. This obviously implies that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is also trivial, but not conversely. A trivial pair is uniquely defined (up to equivalence) by the corresponding generalized module $(\mathfrak{g}, \bar{\mathfrak{g}}/\mathfrak{g})$.

CHAPTER II

CLASSIFICATION OF PAIRS

Preliminaries

1. Let $\mathfrak{g}$ be one of the subalgebras of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ determined in Theorem 1. We assume that the Lie algebra $\mathfrak{g}$ acts naturally on $\mathbb{C}^4$; then $(\mathfrak{g},\mathbb{C}^4)$ is a faithful generalized module. The enumeration of the generalized modules obtained in this way coincide with that of the corresponding subalgebras of $\mathfrak{so}(4,\mathbb{C})$ in Theorem 1.

We say that a pair $(\bar{\mathfrak{g}},\mathfrak{g})$ (a virtual pair $(V,\mathfrak{g})$) has type $(n.m)$, if the corresponding generalized module $(\mathfrak{g},\bar{\mathfrak{g}}/\mathfrak{g})$ (respectively, $(\mathfrak{g},V/\mathfrak{g})$) is isomorphic to the generalized module $n.m$, i.e., to the generalized module $(\mathfrak{g},\mathbb{C}^4)$, where $\mathfrak{g}$ is the subalgebra of $\mathfrak{so}(4,\mathbb{C})$ supplied with the number $n.m$ in Theorem 1.

2. Let $(V,\mathfrak{g})$ be a virtual pair of type $n.m$. Then without loss of generality we can identify the Lie algebra $\mathfrak{g}$ with the subalgebra $n.m$ of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$.

We suppose that $U=\mathbb{C}^4$. Let $\{u_1,u_2,u_3,u_4\}$ be the standard basis of U:

$$u_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\ ,\quad u_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\ ,\quad u_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\ ,\quad u_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\ .$$

3. We define a pair $(\bar{\mathfrak{g}},\mathfrak{g})$ by the commutation table of the Lie algebra $\bar{\mathfrak{g}}$ only. Here by $\{e_1,\dots,e_n,u_1,u_2,u_3,u_4\}$ we denote a basis of $\bar{\mathfrak{g}}$ ($n=\dim\mathfrak{g}$). We assume that the Lie algebra $\mathfrak{g}$ is generated by $e_1,\dots,e_n$.

By p, r, s, etc. we denote the parameters appearing in the process of the classification. If there are some complementary conditions on them, it is indicated just after the table. Otherwise we assume that these parameters run through $\mathbb{C}$.

4. We make use of the following notation:

$\mathcal{D}^n\mathfrak{g}$ are the elements of the derived series of a Lie algebra $\mathfrak{g}$;

$\mathfrak{r}(\mathfrak{g})$ is the radical of $\mathfrak{g}$;

$\mathcal{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$;

$\operatorname{ad}_{\mathfrak{a}} x$, where x is an element of $\mathfrak{g}$, and $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, denotes the restriction of the endomorphism $\operatorname{ad} x$ to $\mathfrak{a}$.

5. In the trivial case $\mathfrak{g}=\{0\}$ the classification of isotropically-faithful pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the classification (up to isomorphism) of all four-dimensional Lie algebras $\bar{\mathfrak{g}}$. It can be found, for example, in [L3].

Proposition 0.1. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 0.1 is equivalent to one and only one of the following pairs:*

$\bar{\mathfrak{g}}_1$	u_1	u_2	u_3	u_4
u_1	0	u_3	u_2	0
u_2	$-u_3$	0	u_1	0
u_3	$-u_2$	$-u_1$	0	0
u_4	0	0	0	0

$\bar{\mathfrak{g}}_2$	u_1	u_2	u_3	u_4
u_1	0	u_3	0	u_1
u_2	$-u_3$	0	0	pu_2
u_3	0	0	0	$(p+1)u_3$
u_4	$-u_1$	$-pu_2$	$-(p+1)u_3$	0

$\bar{\mathfrak{g}}_3$	u_1	u_2	u_3	u_4
u_1	0	0	0	$2u_1$
u_2	0	0	u_1	u_2
u_3	0	$-u_1$	0	u_2+u_3
u_4	$-2u_1$	$-u_2$	$-u_2-u_3$	0

$\bar{\mathfrak{g}}_4$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	u_1+u_2
u_3	0	0	0	u_2+u_3
u_4	$-u_1$	$-u_1-u_2$	$-u_2-u_3$	0

$\bar{\mathfrak{g}}_5$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	u_1+u_2
u_3	0	0	0	pu_3
u_4	$-u_1$	$-u_1-u_2$	$-pu_3$	0

$\bar{\mathfrak{g}}_6$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	pu_2
u_3	0	0	0	ru_3
u_4	$-u_1$	$-pu_2$	$-ru_3$	0

$\bar{\mathfrak{g}}_7$	u_1	u_2	u_3	u_4
u_1	0	0	u_1	0
u_2	0	0	0	u_2
u_3	$-u_1$	0	0	0
u_4	0	$-u_2$	0	0

$\bar{\mathfrak{g}}_8$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_2
u_2	0	0	0	0
u_3	0	0	0	u_1
u_4	$-u_2$	0	$-u_1$	0

$\bar{\mathfrak{g}}_9$	u_1	u_2	u_3	u_4
u_1	0	0	0	u_1
u_2	0	0	0	0
u_3	0	0	0	u_2
u_4	$-u_1$	0	$-u_2$	0

$\bar{\mathfrak{g}}_{10}$	u_1	u_2	u_3	u_4
u_1	0	0	0	0
u_2	0	0	u_1	0
u_3	0	$-u_1$	0	0
u_4	0	0	0	0

$\bar{\mathfrak{g}}_{11}$	u_1	u_2	u_3	u_4
u_1	0	0	0	0
u_2	0	0	0	0
u_3	0	0	0	0
u_4	0	0	0	0

1. One-dimensional case

Proposition 1.1. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 1.1 is equivalent to one and only one of the following pairs:*

$\lambda = 0$

1.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	u_3
u_4	0	0	$-u_2$	$-u_3$	0

2.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	pu_2
u_3	u_3	0	0	0	u_3
u_4	0	0	$-pu_2$	$-u_3$	0

3.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1+u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1-u_2$	0	0	0
u_4	0	0	0	0	0

4.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	u_2	0
u_2	0	0	0	0	0
u_3	u_3	$-u_2$	0	0	0
u_4	0	0	0	0	0

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	u_2
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	$-u_2$	0	0

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	u_3	0	0	0	0
u_4	0	0	$-u_2$	0	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	e_1	0
u_2	0	0	0	0	0
u_3	u_3	$-e_1$	0	0	0
u_4	0	0	0	0	0

$\lambda = \frac{1}{2}$

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	$-2e_1$	u_2
u_2	$-\frac{1}{2}u_2$	0	0	u_4	0
u_3	u_3	$2e_1$	$-u_4$	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
u_1	$-u_1$	0	0	0	u_2
u_2	$-\frac{1}{2}u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	$\frac{1}{2}u_4$	$-u_2$	0	0	0

$|\lambda| < 1,\ -\frac{\pi}{2} < \arg\lambda \leqslant \frac{\pi}{2}$ *or* $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \frac{\pi}{2}$

10.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	λu_2	$-u_3$	$-\lambda u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0
u_3	u_3	0	0	0	0
u_4	λu_4	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1, that is $\mathfrak{h} = \mathfrak{g}$.

Lemma. *Any virtual structure q on generalized module 1.1 is equivalent to one of the following:*

a) $\lambda = 0$

$$C_1(e_1) = (\,0 \quad p \quad 0 \quad r\,);$$

b) $\lambda \neq 0$

$$C_2(e_1) = (\,0 \quad 0 \quad 0 \quad 0\,).$$

Proof. Any virtual structure q has the form:

$$C(e_1) = (\,c_1 \quad c_2 \quad c_3 \quad c_4\,).$$

Suppose $\lambda = 0$. Put

$$H = (\,c_1 \quad 0 \quad -c_3 \quad 0\,)$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then $C_1(x) = (\,0 \quad c_2 \quad 0 \quad c_4\,)$. By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent.

Now suppose $\lambda \neq 0$. Similarly, putting

$$H = (\,c_1 \quad c_2/\lambda \quad -c_3 \quad -c_4/\lambda\,)$$

and $C_2(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$, we see that

$$C_2(e_1) = (\,0 \quad 0 \quad 0 \quad 0\,).$$

By Proposition 4, Chapter I, the virtual structures C and C_2 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda = 0$. The vectors $[e_1, u_i]$, $1 \leqslant i \leqslant 4$, have the form:

$$\begin{aligned} [e_1, u_1] &= u_1, \\ [e_1, u_2] &= pe_1, \\ [e_1, u_3] &= -u_3, \\ [e_1, u_4] &= re_1. \end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_4, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3.$$

Since

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),\\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),\\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),\\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),
\end{aligned}$$

we have

$$\begin{aligned}
&[u_1, u_2] = \alpha u_1,\\
&[u_1, u_3] = ae_1 + \beta_1 u_2 + \beta_2 u_4,\\
&[u_1, u_4] = \gamma u_1,\\
&[u_2, u_3] = \delta u_3,\\
&[u_2, u_4] = be_1 + \eta_1 u_2 + \eta_2 u_4,\\
&[u_3, u_4] = \varepsilon u_3.
\end{aligned}$$

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.1 is equivalent to the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_1) = e_1,\\
&\pi(u_1) = u_1,\\
&\pi(u_2) = \alpha e_1 + u_2,\\
&\pi(u_3) = u_3,\\
&\pi(u_4) = \gamma e_1 + u_4.
\end{aligned}$$

The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ has the form (*):

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	$ae_1+\beta_1u_2+\beta_2u_4$	0
u_2	0	0	0	δu_3	$be_1+\eta_1u_2+\eta_2u_4$
u_3	u_3	$-ae_1-\beta_1u_2-\beta_2u_4$	$-\delta u_3$	0	εu_3
u_4	0	0	$-be_1-\eta_1u_2-\eta_2u_4$	$-\varepsilon u_3$	0

Consider the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of the form (*) .

Let $\varepsilon \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.1 is equivalent to the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ with $\delta = 0$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2 + \frac{\delta}{\varepsilon} u_4, \\ \pi(u_3) &= u_3, \\ \pi(u_4) &= u_4. \end{aligned}$$

The case $\varepsilon = 0$, $\delta \neq 0$ is equivalent to the previous case by means of the mapping $\pi_1 : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi_1(e_1) &= e_1, \\ \pi_1(u_1) &= u_1, \\ \pi_1(u_2) &= u_4, \\ \pi_1(u_3) &= u_3, \\ \pi_1(u_4) &= u_2. \end{aligned}$$

Using the Jacobi identity we obtain: $p = r = b = 0$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	$ae_1 + \beta_1 u_2 + \beta_2 u_4$	0
u_2	0	0	0	0	$\eta_1 u_2 + \eta_2 u_4$
u_3	u_3	$-ae_1 - \beta_1 u_2 - \beta_2 u_4$	0	0	εu_3
u_4	0	0	$-\eta_1 u_2 - \eta_2 u_4$	$-\varepsilon u_3$	0

,

where

$$\begin{cases} \beta_2 \eta_1 = \beta_2 \eta_2 = 0, \\ \varepsilon a = \varepsilon \eta_2 = 0, \\ \beta_1 \eta_1 = \beta_1 \varepsilon, \\ \beta_1 \eta_2 = \beta_2 \varepsilon. \end{cases}$$

1.1°. $\varepsilon \neq 0$. Then we have

$$\begin{cases} \beta_2 = \eta_2 = a = 0, \\ \beta_1(\varepsilon - \eta_1) = 0. \end{cases}$$

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= u_3, \\ \pi(u_4) &= \frac{1}{\varepsilon} u_4. \end{aligned}$$

The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	$\beta_1 u_2$	0
u_2	0	0	0	0	$\eta_1 u_2$
u_3	u_3	$-\beta_1 u_2$	0	0	u_3
u_4	0	0	$-\eta_1 u_2$	$-u_3$	0 ,

where $\beta_1(1-\eta_1)=0$.

1.1.1°. $\beta_1 \neq 0$. Then we have $\eta_1 = 1$. The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(u_1) &= \frac{1}{\beta_1} u_1,\\
\pi(u_2) &= u_2,\\
\pi(u_3) &= u_3,\\
\pi(u_4) &= u_4.
\end{aligned}$$

1.1.2°. $\beta_1 = 0$. The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$.

1.2°. $\varepsilon = 0$.

1.2.1°. $\beta_1 \neq 0$. Then we have $\eta_1 = \eta_2 = 0$.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(u_1) &= u_1,\\
\pi(u_2) &= \beta_1 u_2 + \beta_2 u_4,\\
\pi(u_3) &= u_3,\\
\pi(u_4) &= u_4.
\end{aligned}$$

The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	$ae_1 + u_2$	0
u_2	0	0	0	0	0
u_3	u_3	$-ae_1 - u_2$	0	0	0
u_4	0	0	0	0	0 .

1.2.1.1°. $a \neq 0$. The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = \frac{1}{a}u_1,$$
$$\pi(u_2) = \frac{1}{a}u_2,$$
$$\pi(u_3) = u_3,$$
$$\pi(u_4) = u_4.$$

1.2.1.2°. $a = 0$. The pair $(\bar{\mathfrak{g}'}, \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$.

1.2.2°. $\beta_1 = 0$. The case $\beta_2 \neq 0$ is equivalent to the case 1.2.1° by means of the mapping $\pi_1 : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, where

$$\pi_1(e_1) = -e_1,$$
$$\pi_1(u_1) = u_3,$$
$$\pi_1(u_2) = u_4,$$
$$\pi_1(u_3) = u_1,$$
$$\pi_1(u_4) = u_2.$$

Let $\beta_1 = \beta_2 = 0$.

1.2.2.1°. $\eta_1 \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2 + \frac{\eta_2}{\eta_1}u_4,$$
$$\pi(u_3) = u_3,$$
$$\pi(u_4) = \frac{1}{\eta_1}u_4.$$

The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	0	$-u_3$	0
u_1	$-u_1$	0	0	ae_1	0
u_2	0	0	0	0	u_2
u_3	u_3	$-ae_1$	0	0	0
u_4	0	0	$-u_2$	0	0 .

1.2.2.1.1°. $a \neq 0$. The pair $(\bar{\mathfrak{g}'}, \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = \frac{1}{a}u_1,$$
$$\pi(u_2) = \frac{1}{a}u_2,$$
$$\pi(u_3) = u_3,$$
$$\pi(u_4) = u_4.$$

1.2.2.1.2°. $a = 0$. The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$.

1.2.2.2°. $\eta_1 = 0$. The case $\eta_2 \neq 0$ is equivalent to the case 1.2.2.1° by means of the mapping $\pi_1 : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi_1(e_1) &= -e_1, \\
\pi_1(u_1) &= u_3, \\
\pi_1(u_2) &= u_4, \\
\pi_1(u_3) &= u_1, \\
\pi_1(u_4) &= u_2.
\end{aligned}$$

Let $\eta_1 = \eta_2 = 0$.

1.2.2.2.1°. $a \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= \frac{1}{a} u_1, \\
\pi(u_2) &= \frac{1}{a} u_2, \\
\pi(u_3) &= u_3, \\
\pi(u_4) &= u_4.
\end{aligned}$$

1.2.2.1.2°. $a = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$.

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_5 = 4$, $\dim \mathcal{D}\bar{\mathfrak{g}}_i = 3$, $i \in \{1, 2(p \neq 0), 3, 4, 6, 7\}$, $\dim \mathcal{D}\bar{\mathfrak{g}}_j = 2$, $j \in \{2(p = 0), 10\}$ we see that the pairs $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$, $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ and $(\bar{\mathfrak{g}}_j, \mathfrak{g}_j)$ are not equivalent to each other.

Since $\dim \mathcal{Z}(\bar{\mathfrak{g}}_{10}) = 2$ and $\dim \mathcal{Z}(\bar{\mathfrak{g}}_2\,(p = 0)) = 1$, we see that the pairs $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ $(p = 0)$ are not equivalent.

Since

$$\begin{gathered}
\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 = 1, \quad \mathcal{Z}(\bar{\mathfrak{g}}_1) = \{0\}; \\
\dim \mathcal{D}^2\bar{\mathfrak{g}}_4 = 1, \quad \mathcal{Z}(\bar{\mathfrak{g}}_4) \neq \{0\}; \\
\dim \mathcal{D}^2\bar{\mathfrak{g}}_3 = 3, \quad \mathcal{D}\bar{\mathfrak{g}}_3 \cap \mathfrak{g}_3 = \{0\}; \\
\dim \mathcal{D}^2\bar{\mathfrak{g}}_7 = 3, \quad \mathcal{D}\bar{\mathfrak{g}}_7 \cap \mathfrak{g}_7 \neq \{0\}; \\
\dim \mathcal{D}^2\bar{\mathfrak{g}}_j = 0, \quad j \in \{2, 6\},
\end{gathered}$$

we see that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = j, 1, 3, 4, 7$ are not equivalent to each other.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{C})$, where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i}$, $i = 2, 6$, in the basis $\{u_1, u_2, u_3\}$. We have:

$$f_2(\bar{\mathfrak{g}}_2) = \left\{ \left. \begin{pmatrix} x & 0 & 0 \\ 0 & py & 0 \\ 0 & 0 & y - x \end{pmatrix} \right| x, y \in \mathbb{C} \right\},$$

$$f_6(\bar{\mathfrak{g}}_6) = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & -x \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\}.$$

Since the subalgebras $f_2(\bar{\mathfrak{g}}_2)$ and $f_6(\bar{\mathfrak{g}}_6)$ of the Lie algebra $\mathfrak{gl}(3,\mathbb{C})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ are not equivalent.

Consider the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_2', \mathfrak{g}_2')$ with parameters p and p' respectively.

Since the subalgebras $f_2(\bar{\mathfrak{g}}_2)$ and ${f_2}'(\bar{\mathfrak{g}}_2')$ of the Lie algebra $\mathfrak{gl}(3,\mathbb{C})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_2', \mathfrak{g}_2')$ are not equivalent, whenever $p \neq p'$.

2°. $\lambda = \frac{1}{2}$.

The vectors $[e_1, u_i]$, $1 \leqslant i \leqslant 4$, have the form:

$$\begin{aligned} [e_1, u_1] &= u_1, \\ [e_1, u_2] &= \frac{1}{2} u_2, \\ [e_1, u_3] &= -u_3, \\ [e_1, u_4] &= -\frac{1}{2} u_4. \end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \ \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1, \ \bar{\mathfrak{g}}^{(\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}u_2, \ \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3, \ \bar{\mathfrak{g}}^{(-\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}u_4.$$

Since

$$\begin{aligned} [u_1, u_2] &\in \bar{\mathfrak{g}}^{(\frac{3}{2})}(\mathfrak{h}), \\ [u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\ [u_1, u_4] &\in \bar{\mathfrak{g}}^{(\frac{1}{2})}(\mathfrak{h}), \\ [u_2, u_3] &\in \bar{\mathfrak{g}}^{(-\frac{1}{2})}(\mathfrak{h}), \\ [u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\ [u_3, u_4] &\in \bar{\mathfrak{g}}^{(-\frac{3}{2})}(\mathfrak{h}), \end{aligned}$$

we have

$$\begin{aligned} [u_1, u_2] &= 0, \\ [u_1, u_3] &= a e_1, \\ [u_1, u_4] &= \alpha u_2, \\ [u_2, u_3] &= \beta u_4, \\ [u_2, u_4] &= b e_1, \\ [u_3, u_4] &= 0. \end{aligned}$$

Using the Jacobi identity we obtain: $b = 0,\ a = -2\alpha\beta$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$$\begin{array}{c|ccccc}
[\,,] & e_1 & u_1 & u_2 & u_3 & u_4 \\
\hline
e_1 & 0 & u_1 & \frac{1}{2}u_2 & -u_3 & -\frac{1}{2}u_4 \\
u_1 & -u_1 & 0 & 0 & -2\alpha\beta e_1 & \alpha u_2 \\
u_2 & -\frac{1}{2}u_2 & 0 & 0 & \beta u_4 & 0 \\
u_3 & u_3 & 2\alpha\beta e_1 & -\beta u_4 & 0 & 0 \\
u_4 & \frac{1}{2}u_4 & -\alpha u_2 & 0 & 0 & 0
\end{array}.$$

2.1°. $\alpha\beta \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_8 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= \frac{1}{\alpha}u_1, \\
\pi(u_2) &= u_2, \\
\pi(u_3) &= \frac{1}{\beta}u_3, \\
\pi(u_4) &= u_4.
\end{aligned}$$

2.2°. $\alpha \neq 0,\ \beta = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_9 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= \frac{1}{\alpha}u_1, \\
\pi(u_2) &= u_2, \\
\pi(u_3) &= u_3, \\
\pi(u_4) &= u_4.
\end{aligned}$$

2.3°. $\alpha = 0,\ \beta \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_9 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= -e_1, \\
\pi(u_1) &= -\frac{1}{\beta}u_3, \\
\pi(u_2) &= u_4, \\
\pi(u_3) &= u_1, \\
\pi(u_4) &= u_2.
\end{aligned}$$

2.4°. $\alpha = \beta = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_8 = 5,\ \dim \mathcal{D}^2\bar{\mathfrak{g}}_9 = 1,\ \dim \mathcal{D}^2\bar{\mathfrak{g}}_{10} = 0$ we see that the pairs $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$, $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ and $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$ are not equivalent to each other.

3°. $\lambda = 1$.
The vectors $[e_1, u_i]$, $1 \leqslant i \leqslant 4$, have the form:

$$\begin{aligned}
[e_1, u_1] &= u_1, \\
[e_1, u_2] &= u_2, \\
[e_1, u_3] &= -u_3, \\
[e_1, u_4] &= -u_4.
\end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_2, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3 \oplus \mathbb{C}u_4.$$

Since

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), \\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}),
\end{aligned}$$

we have

$$\begin{aligned}
[u_1, u_2] &= 0, \\
[u_1, u_3] &= ae_1, \\
[u_1, u_4] &= be_1, \\
[u_2, u_3] &= ce_1, \\
[u_2, u_4] &= de_1, \\
[u_3, u_4] &= 0.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = d = 0$, and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$.

4°. $\lambda \notin \{0, \frac{1}{2}, 1\}$.
The vectors $[e_1, u_i]$, $1 \leqslant i \leqslant 4$, have the form:

$$\begin{aligned}
[e_1, u_1] &= u_1, \\
[e_1, u_2] &= \lambda u_2, \\
[e_1, u_3] &= -u_3, \\
[e_1, u_4] &= -\lambda u_4.
\end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1,\ \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1,\ \bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}) = \mathbb{C}u_2,\ \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3,\ \bar{\mathfrak{g}}^{(-\lambda)}(\mathfrak{h}) = \mathbb{C}u_4.$$

Since

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1+\lambda)}(\mathfrak{h}),\\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}),\\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(\lambda-1)}(\mathfrak{h}),\\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(-1-\lambda)}(\mathfrak{h}),
\end{aligned}$$

we have

$$\begin{aligned}
&[u_1, u_2] = 0,\\
&[u_1, u_3] = ae_1,\\
&[u_1, u_4] = 0,\\
&[u_2, u_3] = 0,\\
&[u_2, u_4] = be_1,\\
&[u_3, u_4] = 0.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = 0$, and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$.

Thus the proof of the Proposition is complete.

Proposition 1.2. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.2 is equivalent to the trivial pair:*

1.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	u_1	u_1+u_2	$-u_3-u_4$	$-u_4$
u_1	$-u_1$	0	0	0	0
u_2	$-u_1-u_2$	0	0	0	0
u_3	u_3+u_4	0	0	0	0
u_4	u_4	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1, that is $\mathfrak{h} = \mathfrak{g}$.

Lemma. *Any virtual structure q on generalized module 1.2 is equivalent to the trivial.*

Proof. Any virtual structure q has the form:

$$C(e_1) = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c_1 & c_2 - c_1 & c_4 - c_3 & -c_4 \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then $C_1(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$. By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

The vectors $[e_1, u_i]$, $1 \leqslant i \leqslant 4$, have the form:

$$\begin{aligned}
[e_1, u_1] &= u_1, \\
[e_1, u_2] &= u_1 + u_2, \\
[e_1, u_3] &= -u_3 - u_4, \\
[e_1, u_4] &= -u_4.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h})$$

for all $\alpha \in \mathfrak{h}^*$ (Proposition 7, Chapter I).

Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_2, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3 \oplus \mathbb{C}u_4.$$

Since

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), \\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}),
\end{aligned}$$

we have

$$\begin{aligned}
[u_1, u_2] &= 0, \\
[u_1, u_3] &= ae_1, \\
[u_1, u_4] &= be_1, \\
[u_2, u_3] &= ce_1, \\
[u_2, u_4] &= de_1, \\
[u_3, u_4] &= 0.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = d = 0$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

Thus the proof of the Proposition is complete.

Proposition 1.3. *Any pair* $(\bar{\mathfrak{g}},\mathfrak{g})$ *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	e_1	0	u_1	u_2
u_1	$-e_1$	0	$-\frac{1}{2}u_2$	u_3	$\frac{1}{2}u_4$
u_2	0	$\frac{1}{2}u_2$	0	$\frac{1}{2}u_4$	0
u_3	$-u_1$	$-u_3$	$-\frac{1}{2}u_4$	0	0
u_4	$-u_2$	$-\frac{1}{2}u_4$	0	0	0

2.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\lambda e_1+(\lambda+1)u_1+\lambda u_2$	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$\lambda e_1-(\lambda+1)u_1-\lambda u_2$	0	0	0
u_4	$-u_2$	0	$-u_2$	0	0,

$$|\lambda|<1 \text{ or } |\lambda|=1,\ 0\leqslant \arg\lambda\leqslant\pi$$

3.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	u_1	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$-u_1$	0	0	e_1
u_4	$-u_2$	0	$-u_2$	$-e_1$	0

4.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	x	y
u_2	0	0	0	y	z
u_3	$-u_1$	$-x$	$-y$	0	0
u_4	$-u_2$	$-y$	$-z$	0	0,

where

$$x=\frac{1}{1+\lambda}e_1+\frac{\lambda}{1+\lambda}u_1-\frac{1}{1+\lambda}u_2,$$

$$y=-\frac{1}{1+\lambda}e_1+\frac{1}{1+\lambda}u_1+\frac{1}{1+\lambda}u_2,$$

$$z=-\frac{\lambda}{1+\lambda}e_1+\frac{\lambda}{1+\lambda}u_1+\frac{1+2\lambda}{1+\lambda}u_2,$$

$\lambda\neq -1$

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	u_1	u_2
u_3	$-u_1$	0	$-u_1$	0	$-u_3$
u_4	$-u_2$	0	$-u_2$	u_3	0

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	λu_1	$-\lambda e_1+(\lambda+1)u_2$
u_3	$-u_1$	0	$-\lambda u_1$	0	$-\lambda u_3$
u_4	$-u_2$	0	$\lambda e_1-(\lambda+1)u_2$	λu_3	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	$-u_2$	$-e_1$	0

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	$-u_1$	e_1
u_3	$-u_1$	0	u_1	0	e_1+u_3
u_4	$-u_2$	0	$-e_1$	$-e_1-u_3$	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	μu_1	$-\lambda\mu e_1+(\lambda+\mu)u_2$
u_3	$-u_1$	0	$-\mu u_1$	0	$(1-\mu)u_3$
u_4	$-u_2$	$-u_1$	$\lambda\mu e_1-(\lambda+\mu)u_2$	$(\mu-1)u_3$	0

10.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$\frac{1}{2}u_1$	$-\frac{\lambda}{2}e_1+(\lambda+\frac{1}{2})u_2$
u_3	$-u_1$	0	$-\frac{1}{2}u_1$	0	$e_1+\frac{1}{2}u_3$
u_4	$-u_2$	$-u_1$	$\frac{\lambda}{2}e_1-(\lambda+\frac{1}{2})u_2$	$-e_1-\frac{1}{2}u_3$	0

11.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$(1-\lambda)u_1$	$\lambda(\lambda-1)e_1+u_2$
u_3	$-u_1$	0	$(\lambda-1)u_1$	0	$e_1+\lambda u_3$
u_4	$-u_2$	$-u_1$	$\lambda(1-\lambda)e_1-u_2$	$-e_1-\lambda u_3$	0

$$\lambda \neq \tfrac{1}{2}$$

12.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-e_1+2u_1$	u_2
u_2	0	0	0	u_2	$-e_1+u_1$
u_3	$-u_1$	e_1-2u_1	$-u_2$	0	0
u_4	$-u_2$	$-u_2$	e_1-u_1	0	0

13.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	$-u_1$	$-e_1$	0

14.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	u_1
u_3	$-u_1$	0	0	0	0
u_4	$-u_2$	0	$-u_1$	0	0

15.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	u_1	$-e_1+u_1+2u_2$
u_3	$-u_1$	0	$-u_1$	0	0
u_4	$-u_2$	$-u_1$	$e_1-u_1-2u_2$	0	0

16.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	u_1	u_2-u_1
u_3	$-u_1$	0	$-u_1$	0	$-u_3$
u_4	$-u_2$	0	u_1-u_2	u_3	0

17.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	λu_1	$-\lambda e_1+(1-\lambda)u_1+(1+\lambda)u_2$
u_3	$-u_1$	0	$-\lambda u_1$	0	$(1-\lambda)u_3$
u_4	$-u_2$	$-u_1$	$\lambda e_1+(\lambda-1)u_1-(1+\lambda)u_2$	$(\lambda-1)u_3$	0

$$\lambda \neq 1$$

18.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	$\frac{1}{2}u_1$	$-\frac{1}{2}e_1+\frac{1}{2}u_1+\frac{3}{2}u_2$
u_3	$-u_1$	0	$-\frac{1}{2}u_1$	0	$e_1+\frac{1}{2}u_3$
u_4	$-u_2$	$-u_1$	$\frac{1}{2}e_1-\frac{1}{2}u_1-\frac{3}{2}u_2$	$-e_1-\frac{1}{2}u_3$	0

19.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	u_1
u_2	0	0	0	0	u_1+u_2
u_3	$-u_1$	0	0	0	e_1+u_3
u_4	$-u_2$	$-u_1$	$-u_1-u_2$	$-e_1-u_3$	0

20.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$(1-2\lambda)e_1+2\lambda u_1$	$(2\lambda-1)u_2$
u_2	0	0	0	λu_2	$\frac{2\lambda-1}{2(\lambda-1)}e_1-\frac{1}{2(\lambda-1)}u_1$
u_3	$-u_1$	$(2\lambda-1)e_1-2\lambda u_1$	$-\lambda u_2$	0	$(\lambda-1)u_4$
u_4	$-u_2$	$(1-2\lambda)u_2$	$\frac{1-2\lambda}{2(\lambda-1)}e_1+\frac{1}{2(\lambda-1)}u_1$	$(1-\lambda)u_4$	0

$$\lambda \neq 1$$

21.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\frac{1}{3}e_1+\frac{4}{3}u_1$	$\frac{1}{3}u_2$
u_2	0	0	0	$\frac{2}{3}u_2$	$-\frac{1}{2}e_1+\frac{3}{2}u_1$
u_3	$-u_1$	$\frac{1}{3}e_1-\frac{4}{3}u_1$	$-\frac{2}{3}u_2$	0	$e_1-\frac{1}{3}u_4$
u_4	$-u_2$	$-\frac{1}{3}u_2$	$\frac{1}{2}e_1-\frac{3}{2}u_1$	$\frac{1}{3}u_4-e_1$	0

22.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$2u_1$	$2u_2$
u_2	0	0	0	u_2	$e_1-\frac{1}{2}u_1$
u_3	$-u_1$	$-2u_1$	$-u_2$	0	u_4
u_4	$-u_2$	$-2u_2$	$\frac{1}{2}u_1-e_1$	$-u_4$	0

23.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	x	y
u_2	0	0	0	y	z
u_3	$-u_1$	$-x$	$-y$	0	0
u_4	$-u_2$	$-y$	$-z$	0	0 ,

where

$$\begin{aligned}
x &= \frac{\lambda\mu(\lambda-1)}{\lambda+\mu-\lambda\mu}e_1 + \frac{\lambda^2+\mu-\lambda^2\mu}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda(1-\lambda)}{\lambda+\mu-\lambda\mu}u_2,\\
y &= -\frac{\lambda\mu}{\lambda+\mu-\lambda\mu}e_1 + \frac{\mu}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda}{\lambda+\mu-\lambda\mu}u_2,\\
z &= \frac{\lambda\mu(\mu-1)}{\lambda+\mu-\lambda\mu}e_1 + \frac{\mu(1-\mu)}{\lambda+\mu-\lambda\mu}u_1 + \frac{\lambda+\mu^2-\mu^2\lambda}{\lambda+\mu-\lambda\mu}u_2,
\end{aligned}$$

$$\lambda+\mu-\lambda\mu \neq 0.$$

Two pairs corresponding to parameters (λ_1, μ_1) and (λ_2, μ_2) are equivalent if and only if the points (λ_1, μ_1), $(\lambda_2, \mu_2) \in \mathbb{C}^* \times \mathbb{C}^*$ lie in the same orbit of the action of the symmetric group $\mathfrak{S}_3$ on $\mathbb{C}^* \times \mathbb{C}^*$ generated by the transformations

$$(\lambda, \mu) \to (\mu, \lambda); \qquad (\lambda, \mu) \to (\frac{1}{\lambda}, -\frac{\mu}{\lambda}).$$

24.

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	0	0	0	e_1
u_4	$-u_2$	0	0	$-e_1$	0

25.

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	0	0
u_2	0	0	0	0	0
u_3	$-u_1$	0	0	0	0
u_4	$-u_2$	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 1.3 is equivalent to the following:*

$$C_1(e_1) = (p \quad r \quad 0 \quad 0).$$

Proof. Any virtual structure q has the form:

$$C(e_1) = (c_1 \quad c_2 \quad c_3 \quad c_4).$$

Put

$$H = (c_3 \quad c_4 \quad 0 \quad 0)$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then $C_1(x) = (c_1 \quad c_2 \quad 0 \quad 0)$. By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the virtual structure determined in the Lemma.

Then the vectors $[e_1, u_i]$, $1 \leqslant i \leqslant 4$, have the form:

$$\begin{aligned}
[e_1, u_1] &= pe_1, \\
[e_1, u_2] &= re_1, \\
[e_1, u_3] &= u_1, \\
[e_1, u_4] &= u_2.
\end{aligned}$$

Put

$$\begin{aligned}
[u_1, u_2] &= ae_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4, \\
[u_1, u_3] &= be_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4, \\
[u_1, u_4] &= ce_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3 + \gamma_4 u_4, \\
[u_2, u_3] &= de_1 + \delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 + \delta_4 u_4, \\
[u_2, u_4] &= fe_1 + \eta_1 u_1 + \eta_2 u_2 + \eta_3 u_3 + \eta_4 u_4, \\
[u_3, u_4] &= ke_1 + \varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3 + \varepsilon_4 u_4.
\end{aligned}$$

Let $p \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.3 is equivalent to the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ with $p = 1$, $r = 0$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= \frac{1}{p} u_1, \\
\pi(u_2) &= -\frac{r}{p} u_1 + u_2, \\
\pi(u_3) &= \frac{1}{p} u_3, \\
\pi(u_4) &= -\frac{r}{p} u_3 + u_4.
\end{aligned}$$

The case $p = 0$, $r \neq 0$ is equivalent to the previous case by means of the mapping $\pi_1 : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi_1(e_1) &= e_1, \\
\pi_1(u_1) &= u_2, \\
\pi_1(u_2) &= u_1, \\
\pi_1(u_3) &= u_4, \\
\pi_1(u_4) &= u_3.
\end{aligned}$$

Using the Jacobi identity for triples (e_1, u_i, u_j), $1 \leqslant i, j \leqslant 4$ we obtain that:

$$\begin{aligned}
[u_1, u_2] &= ae_1 - \frac{1}{2}u_2,\\
[u_1, u_3] &= be_1 + \beta_2 u_2 + u_3,\\
[u_1, u_4] &= ce_1 + au_1 + \gamma_2 u_2 + \frac{1}{2}u_4,\\
[u_2, u_3] &= de_1 - au_1 + \delta_2 u_2 + \frac{1}{2}u_4,\\
[u_2, u_4] &= fe_1 + \eta_2 u_2,\\
[u_3, u_4] &= ke_1 + (c-d)u_1 + \varepsilon_2 u_2 + 2au_3 + (\gamma_2 - \delta_2)u_4.
\end{aligned}$$

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 1.3 is equivalent to the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ with $a = c = \gamma_2 = 0$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(u_1) &= -\gamma_2 e_1 + u_1,\\
\pi(u_2) &= 2ae_1 + u_2,\\
\pi(u_3) &= -\gamma_2 u_1 + u_3,\\
\pi(u_4) &= (\frac{2}{3}c - \frac{4}{3}a\gamma_2)e_1 + 2au_1 + u_4.
\end{aligned}$$

Using the Jacobi identity for triples (u_i, u_j, u_k), $1 \leqslant i, j, k \leqslant 4$ we obtain: $d = f = k = \delta_2 = \eta_2 = 0$, $\varepsilon_2 = -\frac{b}{2}$.

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	e_1	0	u_1	u_2
u_1	$-e_1$	0	$-\frac{1}{2}u_2$	$be_1 + \beta_2 u_2 + u_3$	$\frac{1}{2}u_4$
u_2	0	$\frac{1}{2}u_2$	0	$\frac{1}{2}u_4$	0
u_3	$-u_1$	$-be_1 - \beta_2 u_2 - u_3$	$-\frac{1}{2}u_4$	0	$-\frac{b}{2}u_2$
u_4	$-u_2$	$-\frac{1}{2}u_4$	0	$\frac{b}{2}u_2$	0 .

The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(u_1) &= u_1,\\
\pi(u_2) &= u_2,\\
\pi(u_3) &= \frac{b}{2}e_1 + \frac{2}{3}\beta_2 u_2 + u_3,\\
\pi(u_4) &= u_4.
\end{aligned}$$

Let $p = r = 0$.
Using the Jacobi identity for triples (e_1, u_i, u_j), $1 \leqslant i, j \leqslant 4$ we obtain:

$$\begin{aligned}
[u_1, u_2] &= 0,\\
[u_1, u_3] &= be_1 + \beta_1 u_1 + \beta_2 u_2,\\
[u_1, u_4] &= ce_1 + \gamma_1 u_1 + \gamma_2 u_2,\\
[u_2, u_3] &= de_1 + \delta_1 u_1 + \delta_2 u_2,\\
[u_2, u_4] &= fe_1 + \eta_1 u_1 + \eta_2 u_2,\\
[u_3, u_4] &= ke_1 + \varepsilon_1 u_1 + \varepsilon_2 u_2 + (\gamma_1 - \delta_1)u_3 + (\gamma_2 - \delta_2)u_4.
\end{aligned}$$

Suppose

$$V = \mathcal{Z}_{\bar{\mathfrak{g}}}(\mathfrak{g}) \quad \text{and} \quad \mathfrak{a} = \{\operatorname{ad}_V x | x \in \bar{\mathfrak{g}}\}.$$

Then $V = \mathbb{C}e_1 \oplus \mathbb{C}u_1 \oplus \mathbb{C}u_2$ and $\mathfrak{a} = \mathbb{C}(\operatorname{ad}_V u_3) \oplus \mathbb{C}(\operatorname{ad}_V u_4)$ is a two-dimensional subalgebra of the Lie algebra $\mathfrak{gl}(V)$.

Let $W = \mathfrak{g} = \mathbb{C}e_1 \subset V$. The Lie algebra $\bar{\mathfrak{g}}$ can be identified with the Lie algebra $\mathfrak{a} \leftthreetimes V$. Note that the following condition holds:

$$V = W \oplus \mathfrak{a}(W) = \mathbb{C}e_1 \oplus \mathbb{C}(\operatorname{ad}_V u_3(W)) \oplus \mathbb{C}(\operatorname{ad}_V u_4(W)).$$

Conversely, suppose $V = \mathbb{C}^3$ and $\mathfrak{a}$ is a two-dimensional subalgebra of $\mathfrak{gl}(V)$. Let W be a one-dimensional subspace of V such that $V = W \oplus \mathfrak{a}(W)$. Put $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes V$ and $\mathfrak{g} = W$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to some pair of type 1.3.

Therefore there exists a one-to-one correspondence between the set of desired pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the set of pairs $(\mathfrak{a}, W)$, where $\mathfrak{a}$ is a two-dimensional subalgebra of $\mathfrak{gl}(V)$ and W is a one-dimensional subspace of V such that

$$V = W \oplus \mathfrak{a}(W).$$

Lemma 2. *Suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are subalgebras of $\mathfrak{gl}(V)$. Then the Lie algebras $\bar{\mathfrak{g}}_1 = \mathfrak{a}_1 \leftthreetimes V$ and $\bar{\mathfrak{g}}_2 = \mathfrak{a}_2 \leftthreetimes V$ are isomorphic if and only if there exists an endomorphism $\varphi \in GL(V)$ such that $\mathfrak{a}_2 = \varphi\mathfrak{a}_1\varphi^{-1}$.*

Proof. Indeed, suppose there exists a $\varphi \in GL(V)$ such that $\mathfrak{a}_2 = \varphi\mathfrak{a}_1\varphi^{-1}$.
Consider the mapping $f : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_2$ defined by

$$f(x, v) = (\varphi x \varphi^{-1}, \varphi(v)) \text{ for } x \in \mathfrak{a}_1,\ v \in V.$$

It is easy to see that f is an isomorphism of Lie algebras.
The converse statement is obvious.

Lemma 3. *Let $\bar{\mathfrak{g}}_2 = \mathfrak{a}_1 \leftthreetimes V$, $\mathfrak{g}_1 = W_1$, $\bar{\mathfrak{g}}_2 = \mathfrak{a}_2 \leftthreetimes V$, and $\mathfrak{g}_2 = W_2$, where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are subalgebras of $\mathfrak{gl}(V)$, W_1 and W_2 are one-dimensional subspaces of V. Then a necessary and sufficient condition for the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_2)$ to be equivalent is that there exist a $\varphi \in GL(V)$ such that $\mathfrak{a}_2 = \varphi\mathfrak{a}_1\varphi^{-1}$ and $\varphi(W_1) = W_2$. In other* *words, the group $GL(V)$ acts on the set of pairs $(\mathfrak{a}, W)$ and the action is defined by*

$$\varphi : (\mathfrak{a}, W) \to (\varphi \mathfrak{a} \varphi^{-1}, \varphi(W)).$$

Proof. It immediately follows from the previous Lemma.

Let us classify (up to transformations determined before) all pairs $(\mathfrak{a}, W)$.

It is known that any two-dimensional subalgebra $\mathfrak{a}$ of the Lie algebra $\mathfrak{gl}(3, \mathbb{C})$ is equivalent (up to conjugation) to one and only one of the following subalgebras (see [KT]):

$$\mathfrak{a}_1 = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & \lambda x & 0 \\ 0 & 0 & y \end{pmatrix} \middle| x, y \in \mathbb{C}, |\lambda| < 1 \text{ or } |\lambda| = 1,\ 0 \leqslant \arg \lambda \leqslant \pi \right\},$$

$$\mathfrak{a}_2 = \left\{ \begin{pmatrix} x+y & 0 & 0 \\ 0 & \lambda x & 0 \\ 0 & 0 & \mu y \end{pmatrix} \middle| x, y \in \mathbb{C} \right\},$$

Two pairs corresponding to parameters (λ_1, μ_1) and (λ_2, μ_2) are equivalent if and only if the points (λ_1, μ_1), $(\lambda_2, \mu_2) \in \mathbb{C}^* \times \mathbb{C}^*$ lie in the same orbit of the action of the symmetric group $\mathfrak{S}_3$ on $\mathbb{C}^* \times \mathbb{C}^*$ generated by the transformations

$$(\lambda, \mu) \to (\mu, \lambda); \qquad (\lambda, \mu) \to (\frac{1}{\lambda}, -\frac{\mu}{\lambda}).$$

$$\mathfrak{a}_3 = \left\{ \begin{pmatrix} y & 0 & x+\lambda y \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} \middle| x, y \in \mathbb{C}, \lambda \in \mathbb{C} \right\},$$

$$\mathfrak{a}_4 = \left\{ \begin{pmatrix} y & 0 & y \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} \middle| x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_5 = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & y \end{pmatrix} \middle| x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_6 = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ 0 & 0 & \lambda y \end{pmatrix} \middle| x, y \in \mathbb{C}, \lambda \in \mathbb{C} \right\},$$

$$\mathfrak{a}_7 = \left\{ \begin{pmatrix} y & 0 & x \\ 0 & \lambda y & 0 \\ 0 & 0 & \mu y \end{pmatrix} \middle| x, y \in \mathbb{C}, \lambda, \mu \in \mathbb{C} \right\},$$

$$\mathfrak{a}_8 = \left\{ \begin{pmatrix} x & y & x \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \middle| x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_9 = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \middle| x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{10} = \left\{ \begin{pmatrix} 0 & y & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{11} = \left\{ \begin{pmatrix} y & y & x \\ 0 & y & y \\ 0 & 0 & y \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{12} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & y & y \\ 0 & 0 & y \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{13} = \left\{ \begin{pmatrix} y & 0 & x \\ 0 & \lambda y & y \\ 0 & 0 & \lambda y \end{pmatrix} \middle| \, x, y \in \mathbb{C}, \lambda \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{14} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{15} = \left\{ \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & y \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{16} = \left\{ \begin{pmatrix} y & y & x \\ 0 & y & 0 \\ 0 & 0 & \lambda y \end{pmatrix} \middle| \, x, y \in \mathbb{C}, \lambda \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{17} = \left\{ \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\},$$

$$\mathfrak{a}_{18} = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & \lambda x & y \\ 0 & 0 & (2\lambda - 1)x \end{pmatrix} \middle| \, x, y \in \mathbb{C}, \lambda \in \mathbb{C} \backslash \{1\} \right\},$$

$$\mathfrak{a}_{19} = \left\{ \begin{pmatrix} 0 & y & 0 \\ 0 & x & y \\ 0 & 0 & 2x \end{pmatrix} \middle| \, x, y \in \mathbb{C} \right\}.$$

Consider in detail the case when $\mathfrak{a} = \mathfrak{a}_1$.

For a subalgebra $\mathfrak{a} \subset \mathfrak{gl}(3, \mathbb{C})$ by $A(\mathfrak{a})$ denote the following subgroup of $GL(3, \mathbb{C})$:

$$\mathcal{A}(\mathfrak{a}) = \left\{ X \in GL(3, \mathbb{C}) | X \mathfrak{a} X^{-1} = \mathfrak{a} \right\}.$$

Let $\lambda \notin \{-1, 1\}$, then

$$\mathcal{A}(\mathfrak{a}) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, a, b, c \in \mathbb{C}^* \right\}.$$

Any one-dimensional subspace W of V is equivalent (up to the action of elements of $\mathcal{A}(\mathfrak{a})$) to one and only one of the following subspaces:

$$W = \mathbb{C} \begin{pmatrix} i \\ j \\ k \end{pmatrix}, \quad i, j, k \in \{0, 1\}, \quad i^2 + j^2 + k^2 \neq 0.$$

Let $\lambda = -1$, then

$$\mathcal{A}(\mathfrak{a}) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, a,b,c \in \mathbb{C}^* \right\}.$$

Any one-dimensional subspace W of V is equivalent (up to the action of elements of $\mathcal{A}(\mathfrak{a})$) to one and only one of the following subspaces:

$$W = \mathbb{C}\begin{pmatrix} i \\ j \\ k \end{pmatrix}, \quad i,j,k \in \{0,1\}, \quad j = 0 \text{ if } i = 0, \quad i^2 + j^2 + k^2 \neq 0.$$

Note that the condition

$$V = W \oplus \mathfrak{a}(W)$$

holds only for

$$W_1 = \mathbb{C}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Suppose

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $u_1 = \operatorname{ad}_V u_3(e_1)$, $u_2 = \operatorname{ad}_V u_4(e_1)$.
Then

$$u_1 = \begin{pmatrix} 1 \\ \lambda \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $\{e_1, u_1, u_2\}$ is a basis of V, we obtain that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$, corresponding $\mathfrak{a}_1$ has the form:

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	u_2
u_1	0	0	0	$-\lambda e_1+(\lambda+1)u_1+\lambda u_2$	0
u_2	0	0	0	0	u_2
u_3	$-u_1$	$\lambda e_1-(\lambda+1)u_1-\lambda u_2$	0	0	A
u_4	$-u_2$	0	$-u_2$	$-A$	0 ,

where $A = ae_1 + \alpha u_1 + \beta u_2$.

Consider the following cases:

1°. $\lambda \neq 0$. The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2, \\
\pi(u_3) &= (a - \beta)e_1 + u_3, \\
\pi(u_4) &= (\alpha - \frac{a(1+\lambda)}{\lambda})e_1 - \frac{a}{\lambda}u_1 + u_4.
\end{aligned}$$

2°. $\lambda = 0$.

2.1°. $a \neq 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3,\mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= ae_1,\\
\pi(u_1) &= au_1,\\
\pi(u_2) &= au_2,\\
\pi(u_3) &= -\beta e_1 + u_3,\\
\pi(u_4) &= \alpha e_1 + u_4.
\end{aligned}$$

2.2°. $a = 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ (with $\lambda = 0$).

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_2$ $(\lambda = 0) \neq \dim \mathcal{D}\bar{\mathfrak{g}}_3$ we see that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

If $\lambda = 1$, then there is no any one-dimensional subspace W of V such that

$$V = W \oplus \mathfrak{a}(W).$$

In a similar way we obtain the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 4 - 25$.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_i$, $i = 2 - 25$ we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ are not equivalent.

The proof of the Proposition is complete.

Proposition 1.4. *Any pair* $(\bar{\mathfrak{g}},\mathfrak{g})$ *of type* 1.4 *is equivalent to one and only one of the following pairs:*

1.

$$\begin{array}{c|ccccc}
[\,,] & e_1 & u_1 & u_2 & u_3 & u_4\\
\hline
e_1 & 0 & 0 & u_1 & u_2 & e_1\\
u_1 & 0 & 0 & u_1 & u_2 & u_1\\
u_2 & -u_1 & -u_1 & 0 & u_3 & 0\\
u_3 & -u_2 & -u_2 & -u_3 & 0 & -u_3\\
u_4 & -e_1 & -u_1 & 0 & u_3 & 0
\end{array}$$

2.

$$\begin{array}{c|ccccc}
[\,,] & e_1 & u_1 & u_2 & u_3 & u_4\\
\hline
e_1 & 0 & 0 & u_1 & u_2 & e_1\\
u_1 & 0 & 0 & 0 & 0 & pu_1\\
u_2 & -u_1 & 0 & 0 & 0 & (p-1)u_2\\
u_3 & -u_2 & 0 & 0 & 0 & (p-2)u_3\\
u_4 & -e_1 & -pu_1 & (1-p)u_2 & (2-p)u_3 & 0
\end{array}$$

3.

$$\begin{array}{c|ccccc}
[\,,] & e_1 & u_1 & u_2 & u_3 & u_4\\
\hline
e_1 & 0 & 0 & u_1 & u_2 & e_1\\
u_1 & 0 & 0 & 0 & 0 & 2u_1\\
u_2 & -u_1 & 0 & 0 & e_1 & u_2\\
u_3 & -u_2 & 0 & -e_1 & 0 & 0\\
u_4 & -e_1 & -2u_1 & -u_2 & 0 & 0
\end{array}$$

4.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	u_1	u_2	0
u_2	$-u_1$	$-u_1$	0	u_3	0
u_3	$-u_2$	$-u_2$	$-u_3$	0	0
u_4	0	0	0	0	0

5.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	u_1+u_3
u_4	0	$-u_1$	$-u_2$	$-u_1-u_3$	0

6.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	u_2
u_3	$-u_2$	0	0	0	u_3
u_4	0	$-u_1$	$-u_2$	$-u_3$	0

7.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	$re_1+u_2+u_4$	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2-u_4$	0	pu_4
u_4	0	0	0	$-pu_4$	0

8.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	re_1+u_2	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2$	0	pu_4
u_4	0	0	0	$-pu_4$	0

9.

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	$re_1+u_2+u_4$	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2-u_4$	0	u_1-u_4
u_4	0	0	0	u_4-u_1	0

10.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	u_1	0
u_2	$-u_1$	0	0	re_1+u_2	0
u_3	$-u_2$	$-u_1$	$-re_1-u_2$	0	u_1-u_4
u_4	0	0	0	u_4-u_1	0

11.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	re_1+u_4	0
u_3	$-u_2$	0	$-re_1-u_4$	0	u_4
u_4	0	0	0	$-u_4$	0

12.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	re_1	0
u_3	$-u_2$	0	$-re_1$	0	u_4
u_4	0	0	0	$-u_4$	0

13.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1+u_4	0
u_3	$-u_2$	0	$-e_1-u_4$	0	u_1
u_4	0	0	0	$-u_1$	0

14.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	u_4	0
u_3	$-u_2$	0	$-u_4$	0	u_1
u_4	0	0	0	$-u_1$	0

15.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1+u_4	0
u_3	$-u_2$	0	$-e_1-u_4$	0	0
u_4	0	0	0	0	0

16.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	u_4	0
u_3	$-u_2$	0	$-u_4$	0	0
u_4	0	0	0	0	0

17.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1	0
u_3	$-u_2$	0	$-e_1$	0	u_1
u_4	0	0	0	$-u_1$	0

18.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	0	0	0	u_1
u_4	0	0	0	$-u_1$	0

19.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	e_1	0
u_3	$-u_2$	0	$-e_1$	0	0
u_4	0	0	0	0	0

20.

$[,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	0
u_3	$-u_2$	0	0	0	0
u_4	0	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $A(e_1) = 0$, and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 1.4 is equivalent to the following:*

$$C_1(e_1) = (\,p \quad 0 \quad 0 \quad r\,).$$

Proof. Any virtual structure q has the form:

$$C(e_1) = (\,c_1 \quad c_2 \quad c_3 \quad c_4\,).$$

Put

$$H = (\,c_2 \quad c_3 \quad 0 \quad 0\,)$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then $C_1(x) = (\,c_1 \quad 0 \quad 0 \quad c_4\,)$. By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 1.4. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the virtual structure determined in the Lemma.

Then the vectors $[e_1, u_i]$, $1 \leqslant i \leqslant 4$, have the form:

$$\begin{aligned}
[e_1, u_1] &= pe_1,\\
[e_1, u_2] &= u_1,\\
[e_1, u_3] &= u_2,\\
[e_1, u_4] &= re_1.
\end{aligned}$$

Put

$$\begin{aligned}
[u_1, u_2] &= ae_1 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4,\\
[u_1, u_3] &= be_1 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4,\\
[u_1, u_4] &= ce_1 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3 + \gamma_4 u_4,\\
[u_2, u_3] &= de_1 + \delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 + \delta_4 u_4,\\
[u_2, u_4] &= fe_1 + \eta_1 u_1 + \eta_2 u_2 + \eta_3 u_3 + \eta_4 u_4,\\
[u_3, u_4] &= ke_1 + \varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3 + \varepsilon_4 u_4.
\end{aligned}$$

Suppose $r = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	$\alpha_1 u_1$	$\beta_1 u_1 + \alpha_1 u_2$	$\gamma_1 u_1$
u_2	$-u_1$	$-\alpha_1 u_1$	0	A	$\eta_1 u_1 + \gamma_1 u_2$
u_3	$-u_2$	$-\beta_1 u_1 - \alpha_1 u_2$	$-A$	0	B
u_4	0	$-\gamma_1 u_1$	$-\eta_1 u_1 - \gamma_1 u_2$	$-B$	0

,

where

$$\begin{aligned}
A &= de_1 + \delta_1 u_1 + \beta_1 u_2 + \alpha_1 u_3 + \delta_4 u_4,\\
B &= ke_1 + \varepsilon_1 u_1 + \eta_1 u_2 + \gamma_1 u_3 + \varepsilon_4 u_4,
\end{aligned}$$

and

$$\begin{cases} \alpha_1\gamma_1 = 0, \\ 2\alpha_1\beta_1 + \gamma_1\delta_4 = 0, \\ \gamma_1(\beta_1 + \varepsilon_4) = 0, \\ 2d\gamma_1 - k\alpha_1 = 0, \\ 2\gamma_1\delta_4 - \alpha_1\varepsilon_4 = 0, \\ \varepsilon_4\eta_1 - k - 2\alpha_1\varepsilon_1 + \gamma_1\delta_1 = 0. \end{cases}$$

Consider the following cases:

1°. $\alpha_1 \neq 0$. Then we have $k = \beta_1 = \gamma_1 = \varepsilon_1 = \varepsilon_4 = 0$.

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	$\alpha_1 u_1$	$\alpha_1 u_2$	0
u_2	$-u_1$	$-\alpha_1 u_1$	0	A	$\eta_1 u_1$
u_3	$-u_2$	$-\alpha_1 u_2$	$-A$	0	$\eta_1 u_2$
u_4	0	0	$-\eta_1 u_1$	$-\eta_1 u_2$	0 ,

where $A = de_1 + \delta_1 u_1 + \alpha_1 u_3 + \delta_4 u_4$.

The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= \frac{1}{\alpha_1}u_1, \\ \pi(u_2) &= \frac{1}{\alpha_1}u_2, \\ \pi(u_3) &= \frac{1}{\alpha_1}u_3, \\ \pi(u_4) &= \eta_1 e_1 + u_4. \end{aligned}$$

The pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	u_1	u_2	0
u_2	$-u_1$	$-u_1$	0	A	0
u_3	$-u_2$	$-u_2$	$-A$	0	0
u_4	0	0	0	0	0 ,

where $A = de_1 + \delta_1 u_1 + u_3 + \delta_4 u_4$.

Then the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_4,\mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= de_1 + \frac{\delta_1 - d}{2}u_1 + u_3 + \delta_4 u_4, \\ \pi(u_4) &= u_4. \end{aligned}$$

2°. $\alpha_1 = 0$, $\gamma_1 \neq 0$. Then we have

$$\begin{cases} d = \delta_4 = 0, \\ \varepsilon_4 = -\beta_1, \\ k = \delta_1\gamma_1 - \beta_1\eta_1. \end{cases}$$

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	β_1u_1	γ_1u_1
u_2	$-u_1$	0	0	$\delta_1u_1+\beta_1u_2$	$\eta_1u_1+\gamma_1u_2$
u_3	$-u_2$	$-\beta_1u_1$	$-\delta_1u_1-\beta_1u_2$	0	A
u_4	0	$-\gamma_1u_1$	$-\eta_1u_1-\gamma_1u_2$	$-A$	0 ,

where $A = (\delta_1\gamma_1 - \beta_1\eta_1)e_1 + \varepsilon_1u_1 + \eta_1u_2 + \gamma_1u_3 - \beta_1u_4$.

The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = u_3 - \frac{\beta_1}{\gamma_1}u_4,$$
$$\pi(u_4) = \frac{1}{\gamma_1}u_4.$$

The pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ has the form:

$[\,,]$	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	δ_1u_1	$\eta_1u_1+u_2$
u_3	$-u_2$	0	$-\delta_1u_1$	0	A
u_4	0	$-u_1$	$-\eta_1u_1-u_2$	$-A$	0 ,

where $A = \delta_1e_1 + \varepsilon_1u_1 + \eta_1u_2 + u_3$.

2.1°. $\varepsilon_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_1) = \sqrt{\varepsilon_1}e_1,$$
$$\pi(u_1) = \varepsilon_1u_1,$$
$$\pi(u_2) = \sqrt{\varepsilon_1}u_2,$$
$$\pi(u_3) = \delta_1e_1 + u_3,$$
$$\pi(u_4) = \eta_1e_1 + u_4.$$

2.2°. $\varepsilon_1 = 0$. Then the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_6,\mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \delta_1e_1 + u_3,$$
$$\pi(u_4) = \eta_1e_1 + u_4.$$

3°. $\alpha_1 = \gamma_1 = 0$. Then we have $k = \varepsilon_4\eta_1$ and the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	$\beta_1 u_1$	0
u_2	$-u_1$	0	0	A	$\eta_1 u_1$
u_3	$-u_2$	$-\beta_1 u_1$	$-A$	0	B
u_4	0	0	$-\eta_1 u_1$	$-B$	0 ,

where

$$A = de_1 + \delta_1 u_1 + \beta_1 u_2 + \delta_4 u_4,$$
$$B = \varepsilon_4\eta_1 e_1 + \varepsilon_1 u_1 + \eta_1 u_2 + \varepsilon_4 u_4.$$

The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \delta_1 e_1 + u_3,$$
$$\pi(u_4) = \eta_1 e_1 + u_4.$$

The pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ has the form (*):

[,]	e_1	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	0
u_1	0	0	0	$\beta_1 u_1$	0
u_2	$-u_1$	0	0	A	0
u_3	$-u_2$	$-\beta_1 u_1$	$-A$	0	B
u_4	0	0	0	$-B$	0 ,

where

$$A = de_1 + \beta_1 u_2 + \delta_4 u_4,$$
$$B = \varepsilon_1 u_1 + \varepsilon_4 u_4.$$

Note that any pair $(\bar{\mathfrak{g}},\mathfrak{g})$ of the form (*) is uniquely defined by the set of parameters $(d, \beta_1, \delta_4, \varepsilon_1, \varepsilon_4)$.

Lemma 2. *Two pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}',\mathfrak{g}')$ of the form (*) defined by sets of parameters $(d, \beta_1, \delta_4, \varepsilon_1, \varepsilon_4)$ and $(d', {\beta_1}', {\delta_4}', {\varepsilon_1}', {\varepsilon_4}')$, respectively, are equivalent if and only if there exist $a, b_4, c_2 \in \mathbb{C}^*$, $c_1 \in \mathbb{C}$ such that*

$$\begin{cases} \beta_1 = b_4 {\beta_1}', \\ d = {b_4}^2 d', \\ \delta_4 = \frac{a {b_4}^2 {\delta_4}'}{c_2}, \\ \varepsilon_4 = b_4 {\varepsilon_4}', \\ \varepsilon_1 = \frac{1}{a^2}(c_2 {\varepsilon_1}' - c_1 {\varepsilon_4}' - c_1 {\beta_1}'). \end{cases}$$

Proof. Suppose the pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}',\mathfrak{g}')$ are equivalent by means of a mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}'$. Let $H = (h_{ij})_{1 \leqslant i,j \leqslant 5}$ be the matrix of π.

Since $\pi(\mathfrak{g}) = \mathfrak{g}'$, we have $h_{1j} = 0$ whenever $2 \leqslant i \leqslant 5$. Since π is an isomorphism of Lie algebras, we have

$$\pi([x, y]) = [\pi(x), \pi(y)] \text{ for } x, y \in \bar{\mathfrak{g}}. \tag{1}$$

Check this condition for vectors of the basis.

After some calculation we obtain that H has the form:

$$H = \begin{pmatrix} a & 0 & 0 & b_1 & 0 \\ 0 & a^2 b_4 & ab_3 & b_2 & c_1 \\ 0 & 0 & ab_4 & b_3 & 0 \\ 0 & 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & b_5 & c_2 \end{pmatrix}.$$

Check condition (1) for vectors u_1, u_2, u_3, u_4.

1. $\pi([u_1, u_2]) = [\pi(u_1), \pi(u_2)] = 0$.
2. $\pi([u_1, u_3]) = [\pi(u_1), \pi(u_3)] \Rightarrow a^2 b_4 \beta_1 u_1 = a^2 {b_4}^2 {\beta_1}' u_1 \Rightarrow$

$$\beta_1 = b_4 {\beta_1}'.$$

3. $\pi([u_1, u_4]) = [\pi(u_1), \pi(u_4)] = 0$.
4. $\pi([u_2, u_3]) = [\pi(u_2), \pi(u_3)] \Rightarrow ade_1 + (ab_3\beta_1 + c_1\delta_4)u_1 + ab_4\beta_1 u_2 + c_2\delta_4 u_4 = a{b_4}^2 d' e_1 + (ab_3 b_4 {\beta_1}' - ab_1 b_4)u_1 + a{b_4}^2 {\beta_1}' u_2 + a{b_4}^2 {\delta_4}' u_4 \Rightarrow$

$$\begin{cases} d = {b_4}^2 d', \\ \delta_4 = \frac{a{b_4}^2 {\delta_4}'}{c_2}, \\ b_1 = -\frac{c_1\delta_4}{ab_4}. \end{cases}$$

5. $\pi([u_2, u_4]) = [\pi(u_2), \pi(u_4)] = 0$.
6. $\pi([u_3, u_4]) = [\pi(u_3), \pi(u_4)] \Rightarrow (a^2 b_4 \varepsilon_1 + c_1\varepsilon_4)u_1 + c_2\varepsilon_4 u_4 = (b_4 c_2 {\varepsilon_1}' - b_4 c_1 {\beta_1}')u_1 + b_4 c_2 {\varepsilon_4}' u_4 \Rightarrow$

$$\begin{cases} \varepsilon_4 = b_4 {\varepsilon_4}', \\ \varepsilon_1 = \frac{1}{a^2}(c_2{\varepsilon_1}' - c_1{\varepsilon_4}' - c_1{\beta_1}'). \end{cases}$$

So, the classification (up to equivalence) of pairs (*) is reduced to the classification of quintuples $(d, \beta_1, \delta_4, \varepsilon_1, \varepsilon_4)$ up to transformations determined in Lemma 2.

After some calculation, we see that every quintuple is equivalent to one and only one of the following:

$$(d, 1, 1, 0, \varepsilon_4),$$
$$(d, 1, 0, 0, \varepsilon_4),$$

$$\begin{aligned}&(d,1,1,1,-1),\\&(d,1,0,1,-1),\\&(d,0,1,0,1),\\&(d,0,0,0,1),\\&(1,0,1,1,0),\\&(0,0,1,1,0),\\&(1,0,1,0,0),\\&(0,0,1,0,0),\\&(1,0,0,1,0),\\&(0,0,0,1,0),\\&(1,0,0,0,0),\\&(0,0,0,0,0).\end{aligned}$$

The corresponding pairs are $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)$, $7 \leqslant i \leqslant 20$ respectively.

Lemma 3.

Let $r \neq 0$. Then any pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to one and only one of the pairs $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)$, $i = 1-3$.

Proof. The proof is similar to that of the previous Lemma.

Since virtual structures of the case $r = 0$ are trivial, and virtual structures of the case $r \neq 0$ are non-trivial, we see that pairs, corresponding to cases $r = 0$ and $r \neq 0$ are not equivalent.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_4 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_i$, $i = 5-20$ we see that the pairs $(\bar{\mathfrak{g}}_4,\mathfrak{g}_4)$ and $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)$ are not equivalent.

Since $\dim[\mathfrak{g}_i, \mathcal{D}\bar{\mathfrak{g}}_i] \neq \dim[\mathfrak{g}_j, \mathcal{D}\bar{\mathfrak{g}}_j]$, $i = 5,6$, $j = 7-20$ we see that the pairs $(\bar{\mathfrak{g}}_i,\mathfrak{g}_i)$ and $(\bar{\mathfrak{g}}_j,\mathfrak{g}_j)$ are not equivalent.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3,\mathbb{C})$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i}$, $i = 5,6$, in the basis $\{u_1,u_2,u_3\}$. We have:

$$f_5(\bar{\mathfrak{g}}_5) = \left\{ \begin{pmatrix} x & y & x \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \middle| \, x,y \in \mathbb{C} \right\},$$

$$f_6(\bar{\mathfrak{g}}_6) = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} \middle| \, x,y \in \mathbb{C} \right\}.$$

Since the subalgebras $f_5(\bar{\mathfrak{g}}_5)$ and $f_6(\bar{\mathfrak{g}}_6)$ of the Lie algebra $\mathfrak{gl}(3,\mathbb{C})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ and $(\bar{\mathfrak{g}}_6,\mathfrak{g}_6)$ are not equivalent.

This completes the proof of the Proposition.

2. Two-dimensional case

Proposition 2.1. *Any pair* $(\bar{\mathfrak{g}},\mathfrak{g})$ *of type* 2.1 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	e_2
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	$-e_2$	0	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	e_1	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	$-e_1$	0	0	0
u_4	0	u_4	0	0	0	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	0	$-u_3$	0
e_2	0	0	0	u_2	0	$-u_4$
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	0	0	0	0	0
u_4	0	u_4	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then

$$A(e_1) = A(e_2) = 0,$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 2.1 is equivalent to the trivial.*

Proof. Put

$$C(e_i) = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} & c^i_{14} \\ c^i_{21} & c^i_{22} & c^i_{23} & c^i_{24} \end{pmatrix}, \quad i = 1, 2.$$

Let us check condition (3), Chapter I for e_1, e_2.

$$C([e_1,e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have:

$$\begin{pmatrix} -c_{11}^2 & 0 & c_{13}^2 & 0 \\ -c_{21}^2 & 0 & c_{23}^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c_{12}^1 & 0 & -c_{14}^1 \\ 0 & c_{22}^1 & 0 & -c_{24}^1 \end{pmatrix} = 0.$$

It follows that:

$$\begin{cases} c_{11}^2 = c_{13}^2 = c_{21}^2 = c_{23}^2 = 0, \\ c_{12}^1 = c_{14}^1 = c_{22}^1 = c_{24}^1 = 0. \end{cases}$$

So, any virtual structure q on generalized module 2.1 has the form:

$$C(e_1) = \begin{pmatrix} c_{11}^1 & 0 & c_{13}^1 & 0 \\ c_{21}^1 & 0 & c_{23}^1 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & 0 & c_{14}^2 \\ 0 & c_{22}^2 & 0 & c_{24}^2 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c_{11}^1 & c_{12}^2 & -c_{13}^1 & -c_{14}^2 \\ c_{21}^1 & c_{22}^2 & -c_{23}^1 & -c_{24}^2 \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = C_1(e_2) = 0.$$

By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of type 2.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is defined by the trivial virtual structure.

Then

$$\begin{aligned}
&[e_1,e_2] = 0, \\
&[e_1,u_1] = u_1, && [e_2,u_1] = 0, \\
&[e_1,u_2] = 0, && [e_2,u_2] = u_2, \\
&[e_1,u_3] = -u_3, && [e_2,u_3] = 0, \\
&[e_1,u_4] = 0, && [e_2,u_4] = -u_4.
\end{aligned}$$

Put

$$\begin{aligned}
[u_1,u_2] &= a_1e_1 + a_2e_2 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3 + \alpha_4u_4, \\
[u_1,u_3] &= b_1e_1 + b_2e_2 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3 + \beta_4u_4, \\
[u_1,u_4] &= c_1e_1 + c_2e_2 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3 + \gamma_4u_4, \\
[u_2,u_3] &= d_1e_1 + d_2e_2 + \delta_1u_1 + \delta_2u_2 + \delta_3u_3 + \delta_4u_4, \\
[u_2,u_4] &= f_1e_1 + f_2e_2 + \eta_1u_1 + \eta_2u_2 + \eta_3u_3 + \eta_4u_4, \\
[u_3,u_4] &= k_1e_1 + k_2e_2 + \varepsilon_1u_1 + \varepsilon_2u_2 + \varepsilon_3u_3 + \varepsilon_4u_4.
\end{aligned}$$

Using the Jacobi identity we obtain that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$$\begin{array}{c|cccccc}
[\,,\,] & e_1 & e_2 & u_1 & u_2 & u_3 & u_4 \\
\hline
e_1 & 0 & 0 & u_1 & 0 & -u_3 & 0 \\
e_2 & 0 & 0 & 0 & u_2 & 0 & -u_4 \\
u_1 & -u_1 & 0 & 0 & 0 & b_1e_1 & 0 \\
u_2 & 0 & -u_2 & 0 & 0 & 0 & f_2e_2 \\
u_3 & u_3 & 0 & -b_1e_1 & 0 & 0 & 0 \\
u_4 & 0 & u_4 & 0 & -f_2e_2 & 0 & 0
\end{array}.$$

Consider the following cases:

1°. $b_1f_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2, \\
\pi(u_3) &= \frac{1}{b_1}u_3, \\
\pi(u_4) &= \frac{1}{f_2}u_4.
\end{aligned}$$

2°. $b_1 \neq 0,\ f_2 = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2, \\
\pi(u_3) &= \frac{1}{b_1}u_3, \\
\pi(u_4) &= u_4.
\end{aligned}$$

3°. $b_1 = 0,\ f_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= -e_2, \\
\pi(e_2) &= -e_1, \\
\pi(u_1) &= \frac{1}{f_2}u_4, \\
\pi(u_2) &= u_3, \\
\pi(u_3) &= u_2, \\
\pi(u_4) &= u_1.
\end{aligned}$$

4°. $b_1 = f_2 = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_3,\mathfrak{g}_3)$.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 = 6$, $\dim \mathcal{D}\bar{\mathfrak{g}}_2 = 5$, $\dim \mathcal{D}\bar{\mathfrak{g}}_3 = 4$ we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent to each other.

This completes the proof of the Proposition.

Proposition 2.2. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.2 is equivalent to one and only one of the following pairs:*

$\lambda = 0$

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	$-2e_2$
u_1	$-u_1$	0	0	0	u_2	$-u_1$
u_2	0	$-u_1$	0	0	0	u_2
u_3	u_3	u_4	$-u_2$	0	0	$2u_3$
u_4	0	$2e_2$	u_1	$-u_2$	$-2u_3$	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	e_2	u_4	0
u_2	0	$-u_1$	$-e_2$	0	$(p-1)u_3$	pu_4
u_3	u_3	u_4	$-u_4$	$(1-p)u_3$	0	0
u_4	0	0	0	$-pu_4$	0	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	u_3	u_4
u_3	u_3	u_4	0	$-u_3$	0	0
u_4	0	0	0	$-u_4$	0	0

$\lambda = 1$

4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	e_2	0
u_2	$-u_2$	$-u_1$	0	0	e_1	e_2
u_3	u_3	u_4	$-e_2$	$-e_1$	0	0
u_4	u_4	0	0	$-e_2$	0	0

5.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	e_2	0
u_3	u_3	u_4	0	$-e_2$	0	0
u_4	u_4	0	0	0	0	0

$\lambda = -\frac{1}{2}$

6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$\frac{3}{2}e_2$	u_1	$-\frac{1}{2}u_2$	$-u_3$	$\frac{1}{2}u_4$
e_2	$-\frac{3}{2}e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	u_4	0	0
u_2	$\frac{1}{2}u_2$	$-u_1$	$-u_4$	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	$-\frac{1}{2}u_4$	0	0	0	0	0

$|\lambda| < 1$ *or* $|\lambda| = 1,\ 0 \leqslant \arg\lambda \leqslant \pi$

7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$(1-\lambda)e_2$	u_1	λu_2	$-u_3$	$-\lambda u_4$
e_2	$(\lambda-1)e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	λu_4	0	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1-\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 \\ \lambda-1 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 2.2 is equivalent to one of the following:*

a) $\lambda \neq 0,\ \lambda \neq \frac{1}{2}$

$$C_1(e_1) = C_1(e_2) = 0;$$

b) $\lambda = 0$

$$C_2(e_1) = \begin{pmatrix} 0 & -p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r \end{pmatrix};$$

c) $\lambda = \frac{1}{2}$

$$C_3(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \end{pmatrix}, \quad C_3(e_2) = \begin{pmatrix} 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Let q be a virtual structure on generalized module 2.2. Without loss of generality it can be assumed that q is primary. Consider the following cases:

a) $\lambda \neq 0$, $\lambda \neq \frac{1}{2}$. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) \supset \mathbb{C}e_1, \quad \mathfrak{g}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{C}e_2,$$

$$U^{(1)}(\mathfrak{h}) \supset \mathbb{C}u_1, \quad U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{C}u_2, \quad U^{(-1)}(\mathfrak{h}) \supset \mathbb{C}u_3, \quad U^{(-\lambda)}(\mathfrak{h}) \supset \mathbb{C}u_4,$$

we have

$$C(e_1) = C(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, the virtual structure q is equivalent to the trivial. Put $C_1 = C$.

b) $\lambda = 0$. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \mathfrak{g}^{(1)}(\mathfrak{h}) = \mathbb{C}e_2,$$

$$U^{(0)}(\mathfrak{h}) = \mathbb{C}u_2 \oplus \mathbb{C}u_4, \quad U^{(1)}(\mathfrak{h}) = \mathbb{C}u_1, \quad U^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & c_2 & 0 & c_3 \\ c_1 & 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_5 & 0 \\ 0 & c_4 & 0 & c_6 \end{pmatrix}.$$

Let us check condition (3), Chapter I, for $x, y \in \mathcal{E}$.

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have:

$$\begin{pmatrix} 0 & 0 & c_5 & 0 \\ 0 & c_4 & 0 & c_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_4 & 0 & c_6 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -c_5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & -c_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -c_3 & 0 \\ 0 & c_1 & 0 & 0 \end{pmatrix}.$$

It follows that $c_2 = -c_1$, $c_3 = 0$, and the virtual structure q has the form:

$$C(e_1) = \begin{pmatrix} 0 & -c_1 & 0 & 0 \\ c_1 & 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_5 & 0 \\ 0 & c_4 & 0 & c_6 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 & -c_5 \\ c_4 & 0 & 0 & 0 \end{pmatrix},$$

and $C_2(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_2(e_1) = \begin{pmatrix} 0 & -c_1 & 0 & 0 \\ c_1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_6 + c_5 \end{pmatrix}.$$

By Proposition 4, Chapter I, the virtual structures C and C_2 are equivalent.

c) $\lambda = \frac{1}{2}$. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \mathfrak{g}^{(\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}e_2,$$

$$U^{(1)}(\mathfrak{h}) = \mathbb{C}u_1, \quad U^{(\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}u_2, \quad U^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3, \quad U^{(-\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}u_4,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us check condition (3), Chapter I, for e_1 and e_2. We have

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now put $C_3 = C$.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda \neq 0$, $\lambda \neq \frac{1}{2}$. Then

$$\begin{aligned}
&[e_1, e_2] = (1-\lambda)e_2, && \\
&[e_1, u_1] = u_1, && [e_2, u_1] = 0, \\
&[e_1, u_2] = \lambda u_2, && [e_2, u_2] = u_1, \\
&[e_1, u_3] = -u_3, && [e_2, u_3] = -u_4, \\
&[e_1, u_4] = -\lambda u_4, && [e_2, u_4] = 0.
\end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \supset \mathbb{C}e_1, && \bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{C}e_2, \\
&\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \supset \mathbb{C}u_1, && \bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}) \supset \mathbb{C}u_2, \\
&\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \supset \mathbb{C}u_3, && \bar{\mathfrak{g}}^{(-\lambda)}(\mathfrak{h}) \supset \mathbb{C}u_4,
\end{aligned}$$

and

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1+\lambda)}(\mathfrak{h}),\\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}),\\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(\lambda-1)}(\mathfrak{h}),\\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(-1-\lambda)}(\mathfrak{h}).
\end{aligned}$$

1.1°. Suppose $\lambda = 1$. Then

$$\begin{aligned}
&[u_1, u_2] = 0,\\
&[u_1, u_3] = ae_1 + be_2,\\
&[u_1, u_4] = ce_1 + de_2,\\
&[u_2, u_3] = fe_1 + ke_2,\\
&[u_2, u_4] = me_1 + ne_2,\\
&[u_3, u_4] = 0.
\end{aligned}$$

Using the Jacobi identity we obtain:

$$\begin{cases} a = c = 0,\\ d = m = 0,\\ f = n = b. \end{cases}$$

It follows that the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	be_2	0
u_2	$-u_2$	$-u_1$	0	0	$be_1 + ke_2$	be_2
u_3	u_3	u_4	$-be_2$	$-be_1 - ke_2$	0	0
u_4	u_4	0	0	$-be_2$	0	0 .

Consider the following cases:

1.1.1°. $b \neq 0$. Then the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) is equivalent to the pair ($\bar{\mathfrak{g}}_4$,$\mathfrak{g}_4$) by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_1) = e_1,\\
&\pi(e_2) = e_2,\\
&\pi(u_1) = \frac{1}{b}u_1,\\
&\pi(u_2) = -\frac{k}{b^2}u_1 + \frac{1}{b}u_2,\\
&\pi(u_3) = u_3,\\
&\pi(u_4) = u_4.
\end{aligned}$$

1.1.2°. $b = 0,\ k \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(u_1) &= \frac{1}{k}u_1,\\
\pi(u_2) &= \frac{1}{k}u_2,\\
\pi(u_3) &= u_3,\\
\pi(u_4) &= u_4.
\end{aligned}$$

1.1.3°. $b = k = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_4 = 6$, $\dim \mathcal{D}\bar{\mathfrak{g}}_5 = 5$, $\dim \mathcal{D}\bar{\mathfrak{g}}_7 = 4$ we see that the pairs $(\bar{\mathfrak{g}}_4,\mathfrak{g}_4)$, $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ and $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ are not equivalent to each other.

1.2°. Suppose $\lambda = -\frac{1}{2}$. Then

$$\begin{aligned}
[u_1,u_2] &= \alpha u_4,\\
[u_1,u_3] &= a e_1,\\
[u_1,u_4] &= b e_2,\\
[u_2,u_3] &= 0,\\
[u_2,u_4] &= c e_1,\\
[u_3,u_4] &= \beta u_2.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = \beta = 0$. It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$\frac{3}{2}e_2$	u_1	$-\frac{1}{2}u_2$	$-u_3$	$\frac{1}{2}u_4$
e_2	$-\frac{3}{2}e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	αu_4	0	0
u_2	$\frac{1}{2}u_2$	$-u_1$	$-\alpha u_4$	0	0	0
u_3	u_3	u_4	0	0	0	0
u_4	$-\frac{1}{2}u_4$	0	0	0	0	0 .

Consider the following cases:

1.2.1°. $\alpha \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6,\mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(u_1) &= \frac{1}{\alpha}u_1,\\
\pi(u_2) &= \frac{1}{\alpha}u_2,\\
\pi(u_3) &= \frac{1}{\alpha}u_3,\\
\pi(u_4) &= \frac{1}{\alpha}u_4.
\end{aligned}$$

1.2.2°. $\alpha = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$.

Since $\dim[\mathcal{D}^2\bar{\mathfrak{g}}_6, \mathcal{D}\bar{\mathfrak{g}}_6] \neq \dim[\mathcal{D}^2\bar{\mathfrak{g}}_7, \mathcal{D}\bar{\mathfrak{g}}_7]$, we see that the pairs $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ and $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ are not equivalent.

1.3°. Suppose $\lambda = -1$. Then

$$\begin{aligned}
[u_1, u_2] &= ae_1, \\
[u_1, u_3] &= be_1, \\
[u_1, u_4] &= ce_2, \\
[u_2, u_3] &= 0, \\
[u_2, u_4] &= de_1, \\
[u_3, u_4] &= fe_1.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = d = f = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$.

1.4°. Suppose $\lambda \notin \{-1, -\frac{1}{2}, 1\}$. Then

$$\begin{aligned}
[u_1, u_2] &= 0, \\
[u_1, u_3] &= ae_1, \\
[u_1, u_4] &= be_2, \\
[u_2, u_3] &= 0, \\
[u_2, u_4] &= ce_1, \\
[u_3, u_4] &= 0.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$.

2°. $\lambda = 0$. Then

$$\begin{aligned}
&[e_1, e_2] = e_2, \\
&[e_1, u_1] = pe_2 + u_1, && [e_2, u_1] = 0, \\
&[e_1, u_2] = -pe_1, && [e_2, u_2] = u_1, \\
&[e_1, u_3] = -u_3, && [e_2, u_3] = -u_4, \\
&[e_1, u_4] = 0, && [e_2, u_4] = re_2.
\end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_4, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}e_2 \oplus \mathbb{C}u_1, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3.$$

Since

$$
\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),\\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),\\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),\\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),
\end{aligned}
$$

we have

$$
\begin{aligned}
[u_1, u_2] &= ae_2 + \alpha u_1,\\
[u_1, u_3] &= be_1 + \beta_1 u_2 + \beta_2 u_4,\\
[u_1, u_4] &= ce_2 + \gamma u_1,\\
[u_2, u_3] &= \delta u_3,\\
[u_2, u_4] &= fe_1 + \eta_1 u_2 + \eta_2 u_4,\\
[u_3, u_4] &= \varepsilon u_3.
\end{aligned}
$$

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.2 is equivalent to the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ with $\alpha = 0$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$, where

$$
\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(u_1) &= -\frac{\alpha}{2} e_2 + u_1,\\
\pi(u_2) &= \frac{\alpha}{2} e_1 + u_2,\\
\pi(u_3) &= u_3,\\
\pi(u_4) &= u_4.
\end{aligned}
$$

Let $\beta_1 \neq 0$. Using the Jacobi identity we obtain:

$$
\begin{cases}
p = 0,\\
r = -2\beta_1,\\
a = \beta_2^2,\\
b = -3\beta_1\beta_2,\\
c = -\beta_1\beta_2,\\
\gamma = -\beta_1,\\
\delta = \beta_2,\\
f = -3\beta_1\beta_2,\\
\eta_1 = \beta_1,\\
\eta_2 = 2\beta_2,\\
\varepsilon = 2\beta_1.
\end{cases}
$$

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	$-2\beta_1 e_2$
u_1	$-u_1$	0	0	$\beta_2^2 e_2$	A	$-\beta_1\beta_2 e_2-\beta_1 u_1$
u_2	0	$-u_1$	$-\beta_2^2 e_2$	0	$\beta_2 u_3$	B
u_3	u_3	u_4	$-A$	$-\beta_2 u_3$	0	$2\beta_1 u_3$
u_4	0	$2\beta_1 e_2$	$\beta_1\beta_2 e_2+\beta_1 u_1$	$-B$	$-2\beta_1 u_3$	0

,

where

$$\begin{cases} A=-3\beta_1\beta_2 e_1+\beta_1 u_2+\beta_2 u_4, \\ B=-3\beta_1\beta_2 e_1+\beta_1 u_2+2\beta_2 u_4. \end{cases}$$

The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ by means of the mapping $\pi:\bar{\mathfrak{g}}'\to\bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1)&=e_1,\\ \pi(e_2)&=e_2,\\ \pi(u_1)&=\frac{1}{\beta_1}u_1,\\ \pi(u_2)&=\frac{1}{\beta_1}u_2,\\ \pi(u_3)&=\frac{1}{\beta_1}u_3,\\ \pi(u_4)&=\frac{1}{\beta_1}u_4. \end{aligned}$$

The pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	$-2e_2$
u_1	$-u_1$	0	0	$\beta_2^2 e_2$	A	$-\beta_2 e_2-u_1$
u_2	0	$-u_1$	$-\beta_2^2 e_2$	0	$\beta_2 u_3$	B
u_3	u_3	u_4	$-A$	$-\beta_2 u_3$	0	$2u_3$
u_4	0	$2e_2$	$\beta_2 e_2+u_1$	$-B$	$-2u_3$	0

,

where

$$\begin{cases} A=-3\beta_2 e_1+u_2+\beta_2 u_4, \\ B=-3\beta_2 e_1+u_2+2\beta_2 u_4. \end{cases}$$

Then the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi:\bar{\mathfrak{g}}_1\to\bar{\mathfrak{g}}'$, where

$$\begin{aligned} \pi(e_1)&=e_1,\\ \pi(e_2)&=e_2,\\ \pi(u_1)&=-\beta_2 e_2+u_1,\\ \pi(u_2)&=-3\beta_2 e_1+u_2+2\beta_2 u_4,\\ \pi(u_3)&=u_3,\\ \pi(u_4)&=u_4. \end{aligned}$$

Let $\beta_1 = 0$. Using the Jacobi identity we obtain:

$$\begin{cases} p = r = b = c = 0, \\ \gamma = f = \eta_1 = \varepsilon = 0, \\ a = {\beta_2}^2, \\ \delta = \eta_2 - \beta_2. \end{cases}$$

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	${\beta_2}^2 e_2$	$\beta_2 u_4$	0
u_2	0	$-u_1$	$-{\beta_2}^2 e_2$	0	$(\eta_2-\beta_2)u_3$	$\eta_2 u_4$
u_3	u_3	u_4	$-\beta_2 u_4$	$(\beta_2-\eta_2)u_3$	0	0
u_4	0	0	0	$-\eta_2 u_4$	0	0 .

Consider the following cases:

2.1°. $\beta_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(e_2) &= e_2, \\ \pi(u_1) &= \frac{1}{\beta_2}u_1, \\ \pi(u_2) &= \frac{1}{\beta_2}u_2, \\ \pi(u_3) &= \frac{1}{\beta_2}u_3, \\ \pi(u_4) &= \frac{1}{\beta_2}u_4. \end{aligned}$$

2.2°. $\beta_2 = 0$, $\eta_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3,\mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(e_2) &= e_2, \\ \pi(u_1) &= \frac{1}{\eta_2}u_1, \\ \pi(u_2) &= \frac{1}{\eta_2}u_2, \\ \pi(u_3) &= \frac{1}{\eta_2}u_3, \\ \pi(u_4) &= \frac{1}{\eta_2}u_4. \end{aligned}$$

2.3°. $\beta_2 = \eta_2 = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$.

It remains to show that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 2, 3, 7$, are not equivalent to each other.

Consider the algebras

$$\tilde{\mathfrak{g}}_i = \bar{\mathfrak{g}}_i / \mathcal{D}^2 \bar{\mathfrak{g}}_i, i = 2, 3, 7$$

and the homomorphisms

$$f_i : \tilde{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{C}) \ (i = 2, 3, 7),$$

where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{\mathcal{D}\tilde{\mathfrak{g}}_i} x$ in the basis $\{e_2 + \mathbb{C}u_4, u_1 + \mathbb{C}u_4, u_3 + \mathbb{C}u_4\}$, $x \in \tilde{\mathfrak{g}}_i$. We have:

$$f_2(\tilde{\mathfrak{g}}_2) = \left\{ \left. \begin{pmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & (1-p)y - x \end{pmatrix} \right| x, y \in \mathbb{C} \right\},$$

$$f_3(\tilde{\mathfrak{g}}_3) = \left\{ \left. \begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ 0 & 0 & -x-y \end{pmatrix} \right| x, y \in \mathbb{C} \right\},$$

$$f_7(\tilde{\mathfrak{g}}_7) = \left\{ \left. \begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ 0 & 0 & -x \end{pmatrix} \right| x, y \in \mathbb{C} \right\}.$$

Since the subalgebras $f_2(\tilde{\mathfrak{g}}_2)$, $f_3(\tilde{\mathfrak{g}}_3)$, and $f_7(\tilde{\mathfrak{g}}_7)$ of the Lie algebra $\mathfrak{gl}(3,\mathbb{C})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$, and $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ are not equivalent to each other.

Consider the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_2', \mathfrak{g}_2')$ with parameters p and p' respectively.

Since the subalgebras $f_2(\tilde{\mathfrak{g}}_2)$ and $f_2{}'(\tilde{\mathfrak{g}}_2')$ of the Lie algebra $\mathfrak{gl}(3,\mathbb{C})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_2', \mathfrak{g}_2')$ are not equivalent, whenever $p \neq p'$.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_i$, $i = 2, 3, 7$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ are not equivalent.

3°. $\lambda = \frac{1}{2}$. Then

$$\begin{aligned}
&[e_1, e_2] = \frac{1}{2} e_2, && \\
&[e_1, u_1] = u_1, && [e_2, u_1] = 0, \\
&[e_1, u_2] = p e_2 + \frac{1}{2} u_2, && [e_2, u_2] = u_1, \\
&[e_1, u_3] = -u_3, && [e_2, u_3] = -u_4, \\
&[e_1, u_4] = -\frac{1}{2} u_4, && [e_2, u_4] = r e_1.
\end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \bar{\mathfrak{g}}^{(\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}e_2 \oplus \mathbb{C}u_2,$$

$$\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3, \quad \bar{\mathfrak{g}}^{(-\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}u_4.$$

Since

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(\frac{3}{2})}(\mathfrak{h}), \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\ &[u_1, u_4] \in \bar{\mathfrak{g}}^{(\frac{1}{2})}(\mathfrak{h}), \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(-\frac{1}{2})}(\mathfrak{h}), \\ &[u_2, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\ &[u_3, u_4] \in \bar{\mathfrak{g}}^{(-\frac{3}{2})}(\mathfrak{h}), \end{aligned}$$

we have

$$\begin{aligned} &[u_1, u_2] = 0, \\ &[u_1, u_3] = ae_1, \\ &[u_1, u_4] = be_2 + \alpha u_2, \\ &[u_2, u_3] = \beta u_4, \\ &[u_2, u_4] = ce_1, \\ &[u_3, u_4] = 0. \end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = p = r = \alpha = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$\frac{1}{2}e_2$	u_1	$\frac{1}{2}u_2$	$-u_3$	$-\frac{1}{2}u_4$
e_2	$-\frac{1}{2}e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{1}{2}u_2$	$-u_1$	0	0	βu_4	0
u_3	u_3	u_4	0	$-\beta u_4$	0	0
u_4	$\frac{1}{2}u_4$	0	0	0	0	0

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \longrightarrow \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, \\ &\pi(u_1) = u_1, \\ &\pi(u_2) = \beta e_2 + u_2, \\ &\pi(u_3) = u_3, \\ &\pi(u_4) = u_4. \end{aligned}$$

This completes the proof of the Proposition.

Proposition 2.3. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 2.3 is equivalent to the trivial pair:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	u_1	$-u_2$	$-u_2-u_3$	u_1+u_4
e_2	$-2e_2$	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0
u_3	u_2+u_3	u_4	0	0	0	0
u_4	$-u_1-u_4$	0	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 2.3 is equivalent to the trivial.*

Proof. Let q be a virtual structure on generalized module 2.3. By Proposition 6, Chapter 1, without loss of generality it can be assumed that q is primary.

Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \mathfrak{g}^{(2)}(\mathfrak{h}) = \mathbb{C}e_2,$$
$$U^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_4, \quad U^{(-1)}(\mathfrak{h}) = \mathbb{C}u_2 \oplus \mathbb{C}u_3,$$

we have

$$C(e_1) = C(e_2) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Then

$$\begin{aligned}
&[e_1, e_2] = 2e_2, && \\
&[e_1, u_1] = u_1, && [e_2, u_1] = 0, \\
&[e_1, u_2] = -u_2, && [e_2, u_2] = u_1, \\
&[e_1, u_3] = -u_2 - u_3, && [e_2, u_3] = -u_4, \\
&[e_1, u_4] = u_1 + u_4, && [e_2, u_4] = 0.
\end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{C}e_2,$$
$$\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_4, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_2 \oplus \mathbb{C}u_3.$$

Since

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}),\\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}),\\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),
\end{aligned}$$

we have

$$\begin{aligned}
[u_1, u_2] &= ae_1,\\
[u_1, u_3] &= be_1,\\
[u_1, u_4] &= ce_2,\\
[u_2, u_3] &= 0,\\
[u_2, u_4] &= de_1,\\
[u_3, u_4] &= fe_1.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = d = f = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

This completes the proof of the Proposition.

Proposition 2.4. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type* 2.4 *is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	u_1	u_2	0
u_2	0	$-u_1$	$-u_1$	0	u_3	0
u_3	u_3	$-u_2$	$-u_2$	$-u_3$	0	0
u_4	0	0	0	0	0	0

2.

[,]	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	0	0	u_1
u_2	0	$-u_1$	0	0	0	u_2
u_3	u_3	$-u_2$	0	0	0	u_3
u_4	0	0	$-u_1$	$-u_2$	$-u_3$	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	0	0
u_3	u_3	$-u_2$	0	0	0	0
u_4	0	0	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 2.4 is equivalent to the trivial.*

Proof. Let q be a virtual structure on generalized module 2.4. Without loss of generality it can be assumed that q is primary.

Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \mathfrak{g}^{(1)}(\mathfrak{h}) = \mathbb{C}e_2,$$

$$U^{(1)}(\mathfrak{h}) = \mathbb{C}u_1, \quad U^{(0)}(\mathfrak{h}) = \mathbb{C}u_2 \oplus \mathbb{C}u_4, \quad U^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & c_2 & 0 & c_3 \\ c_1 & 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_5 & 0 \\ 0 & c_4 & 0 & c_6 \end{pmatrix}.$$

Let us check condition (3), Chapter I, for $x, y \in \mathcal{E}$.

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have:

$$\begin{pmatrix} 0 & 0 & c_5 & 0 \\ 0 & c_4 & 0 & c_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_4 & 0 & c_6 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -c_5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & -c_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c_2 & 0 \\ 0 & c_1 & 0 & 0 \end{pmatrix}.$$

It follows that $c_1 = c_2 = c_3 = 0$, and the virtual structure q has the form:

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_5 & 0 \\ 0 & c_4 & 0 & c_6 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & -c_5 & 0 & c_6 \\ c_4 + c_5 & 0 & 0 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = C_1(e_2) = 0.$$

By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 2.4. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Then

$$\begin{aligned} &[e_1, e_2] = e_2, \\ &[e_1, u_1] = u_1, && [e_2, u_1] = 0, \\ &[e_1, u_2] = 0, && [e_2, u_2] = u_1, \\ &[e_1, u_3] = -u_3, && [e_2, u_3] = u_2, \\ &[e_1, u_4] = 0, && [e_2, u_4] = 0. \end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\begin{aligned} \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) &= \mathbb{C}e_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_4, \\ \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) &= \mathbb{C}e_2 \oplus \mathbb{C}u_1, \\ \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) &= \mathbb{C}u_3. \end{aligned}$$

Since

$$\begin{aligned} &[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \\ &[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\ &[u_1, u_4] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \\ &[u_2, u_3] \in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}), \\ &[u_2, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\ &[u_3, u_4] \in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}), \end{aligned}$$

we have

$$\begin{aligned}
[u_1, u_2] &= ae_2 + \alpha u_1, \\
[u_1, u_3] &= be_1 + \beta_1 u_2 + \beta_2 u_4, \\
[u_1, u_4] &= ce_2 + \gamma u_1, \\
[u_2, u_3] &= \delta u_3, \\
[u_2, u_4] &= fe_1 + \eta_1 u_2 + \eta_2 u_4, \\
[u_3, u_4] &= \varepsilon u_3.
\end{aligned}$$

Using the Jacobi identity we obtain:

$$\begin{cases}
a = b = c = 0, \\
f = \beta_2 = \eta_2 = 0, \\
\beta_1 = \delta = \alpha, \\
\eta_1 = \varepsilon = \gamma.
\end{cases}$$

It follows that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	e_2	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	u_1	u_2	0
u_1	$-u_1$	0	0	αu_1	αu_2	γu_1
u_2	0	$-u_1$	$-\alpha u_1$	0	αu_3	γu_2
u_3	u_3	$-u_2$	$-\alpha u_2$	$-\alpha u_3$	0	γu_3
u_4	0	0	$-\gamma u_1$	$-\gamma u_2$	$-\gamma u_3$	0 ,

where $\alpha\gamma = 0$.

Consider the following cases:

1°. $\alpha \neq 0,\ \gamma = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(u_1) &= \frac{1}{\alpha} u_1, \\
\pi(u_2) &= \frac{1}{\alpha} u_2, \\
\pi(u_3) &= \frac{1}{\alpha} u_3, \\
\pi(u_4) &= \frac{1}{\alpha} u_4.
\end{aligned}$$

2°. $\alpha = 0,\ \gamma \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}\pi(e_1) &= e_1,\\ \pi(e_2) &= e_2,\\ \pi(u_1) &= \frac{1}{\gamma}u_1,\\ \pi(u_2) &= \frac{1}{\gamma}u_2,\\ \pi(u_3) &= \frac{1}{\gamma}u_3,\\ \pi(u_4) &= \frac{1}{\gamma}u_4.\end{aligned}$$

3°. $\alpha = \gamma = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_3,\mathfrak{g}_3)$. Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 = 3$, $\dim \mathcal{D}^2\bar{\mathfrak{g}}_2 = \dim \mathcal{D}^2\bar{\mathfrak{g}}_3 = 2$, but $\dim[\mathcal{D}^2\bar{\mathfrak{g}}_2, \bar{\mathfrak{g}}_2] \neq \dim[\mathcal{D}^2\bar{\mathfrak{g}}_3, \bar{\mathfrak{g}}_3]$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent to each other.

This completes the proof of the Proposition.

Proposition 2.5. *Any pair* $(\bar{\mathfrak{g}},\mathfrak{g})$ *of type* 2.5 *is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	$-2e_1$
e_2	0	0	0	$-2e_2$	$-u_2$	u_1
u_1	0	0	0	$2e_2-u_1$	u_2+u_4	$2e_1-u_1$
u_2	$-u_1$	$2e_2$	u_1-2e_2	0	$-2u_3$	u_2-u_4
u_3	u_4	u_2	$-u_2-u_4$	$2u_3$	0	$2u_3$
u_4	$2e_1$	$-u_1$	u_1-2e_1	u_4-u_2	$-2u_3$	0

2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	$-2e_2$	$-u_2$	u_1
u_1	0	0	0	$-u_1$	u_4	0
u_2	$-u_1$	$2e_2$	u_1	0	$-2u_3$	$-u_4$
u_3	u_4	u_2	$-u_4$	$2u_3$	0	0
u_4	0	$-u_1$	0	u_4	0	0

3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	$e_1+ge_2+(1-h)u_2$	hu_1
u_3	u_4	u_2	$-u_1$	$-e_1-ge_2+(h-1)u_2$	0	$-(g+h)e_1+ke_2-(1+h)u_4$
u_4	0	$-u_1$	0	$-hu_1$	$(g+h)e_1-ke_2+(1+h)u_4$	0

$\mathrm{Re}\, h > 0$ *or* $\mathrm{Re}\, h = 0$, $\mathrm{Im}\, h \geqslant 0$ (*if* $k \neq 0$), $h \in \mathbb{C}$ (*if* $k = 0$)

4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	u_1	0
u_2	$-u_1$	0	0	0	$ge_2+(1-h)u_2$	hu_1
u_3	u_4	u_2	$-u_1$	$-ge_2+(h-1)u_2$	0	$-(g+h)e_1-(1+h)u_4$
u_4	0	$-u_1$	0	$-hu_1$	$(g+h)e_1+(1+h)u_4$	0

$$\operatorname{Re} h > 0 \text{ or } \operatorname{Re} h = 0,\ \operatorname{Im} h \geqslant 0$$

5.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	$e_1+ge_2-u_2$	u_1
u_3	u_4	u_2	0	$-e_1-ge_2+u_2$	0	$-ge_1+ke_2-u_4$
u_4	0	$-u_1$	0	$-u_1$	$ge_1-ke_2+u_4$	0

6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	ge_2-u_2	u_1
u_3	u_4	u_2	0	$-ge_2+u_2$	0	$-ge_1-u_4$
u_4	0	$-u_1$	0	$-u_1$	ge_1+u_4	0

7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1+e_2	0
u_3	u_4	u_2	0	$-e_1-e_2$	0	$-e_1+ke_2$
u_4	0	$-u_1$	0	0	e_1-ke_2	0

8.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_2	0
u_3	u_4	u_2	0	$-e_2$	0	$-e_1$
u_4	0	$-u_1$	0	0	e_1	0

9.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	e_2
u_4	0	$-u_1$	0	0	$-e_2$	0

10.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1	0
u_3	u_4	u_2	0	$-e_1$	0	0
u_4	0	$-u_1$	0	0	0	0

11.

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	0
e_2	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	u_4	u_2	0	0	0	0
u_4	0	$-u_1$	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = A(e_2) = 0,$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 2.5 is equivalent to the following:*

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 & p \\ 0 & 0 & 0 & r \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & s & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix}.$$

Proof. Put

$$C(e_i) = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} & c^i_{14} \\ c^i_{21} & c^i_{22} & c^i_{23} & c^i_{24} \end{pmatrix}, \quad i = 1, 2.$$

Let us check condition (3), Chapter I for e_1, e_2.

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2).$$

We have:

$$\begin{pmatrix} 0 & -c_{11}^2 & c_{14}^2 & 0 \\ 0 & -c_{21}^2 & c_{24}^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -c_{12}^1 & c_{11}^1 \\ 0 & 0 & -c_{22}^1 & c_{21}^1 \end{pmatrix} = 0.$$

We obtain:

$$\begin{cases} c_{11}^2 = c_{21}^2 = 0, \\ c_{11}^1 = c_{21}^1 = 0, \\ c_{14}^2 = c_{12}^1, \\ c_{24}^2 = c_{22}^1. \end{cases}$$

So, any virtual structure q on generalized module 2.5 has the form:

$$C(e_1) = \begin{pmatrix} 0 & c_{12}^1 & c_{13}^1 & c_{14}^1 \\ 0 & c_{22}^1 & c_{23}^1 & c_{24}^1 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & c_{13}^2 & c_{12}^1 \\ 0 & c_{22}^2 & c_{23}^2 & c_{22}^1 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c_{12}^1 & -c_{13}^2 & 0 & -c_{13}^1 \\ c_{22}^1 & -c_{23}^2 & 0 & -c_{23}^1 \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 & c_{14}^1 \\ 0 & 0 & 0 & c_{24}^1 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & 0 & 0 \\ 0 & c_{22}^2 & 0 & 0 \end{pmatrix}.$$

By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of type 2.5. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is defined by the virtual structure determined in the Lemma.
Then

$$\begin{aligned} &[e_1, e_2] = 0, && \\ &[e_1, u_1] = 0, && [e_2, u_1] = 0, \\ &[e_1, u_2] = u_1, && [e_2, u_2] = se_1 + te_2, \\ &[e_1, u_3] = -u_4, && [e_2, u_3] = -u_2, \\ &[e_1, u_4] = pe_1 + re_2, && [e_2, u_4] = u_1. \end{aligned}$$

Put

$$\begin{aligned} [u_1, u_2] &= a_1e_1 + a_2e_2 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3 + \alpha_4u_4, \\ [u_1, u_3] &= b_1e_1 + b_2e_2 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3 + \beta_4u_4, \\ [u_1, u_4] &= c_1e_1 + c_2e_2 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3 + \gamma_4u_4, \\ [u_2, u_3] &= d_1e_1 + d_2e_2 + \delta_1u_1 + \delta_2u_2 + \delta_3u_3 + \delta_4u_4, \\ [u_2, u_4] &= f_1e_1 + f_2e_2 + \eta_1u_1 + \eta_2u_2 + \eta_3u_3 + \eta_4u_4, \\ [u_3, u_4] &= k_1e_1 + k_2e_2 + \varepsilon_1u_1 + \varepsilon_2u_2 + \varepsilon_3u_3 + \varepsilon_4u_4. \end{aligned}$$

Using the Jacobi identity we obtain that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	u_1	u_2	u_3	u_4
e_1	0	0	0	u_1	$-u_4$	$-2\beta_2 e_1$
e_2	0	0	0	$-2\beta_4 e_2$	$-u_2$	u_1
u_1	0	0	0	$2\beta_2\beta_4 e_2 - \beta_4 u_1$	A	$2\beta_2\beta_4 e_1 - \beta_2 u_1$
u_2	$-u_1$	$2\beta_4 e_2$	$\beta_4 u_1 - 2\beta_2\beta_4 e_2$	0	B	C
u_3	u_4	u_2	$-A$	$-B$	0	D
u_4	$2\beta_2 e_1$	$-u_1$	$\beta_2 u_1 - 2\beta_2\beta_4 e_1$	$-C$	$-D$	0

,

where

$$
\begin{aligned}
A &= \beta_1 u_1 + \beta_2 u_2 + \beta_4 u_4,\\
B &= d_1 e_1 + d_2 e_2 + \delta_1 u_1 + (\beta_1 - \eta_1) u_2 - 2\beta_4 u_3,\\
C &= \eta_1 u_1 + \beta_2 u_2 - \beta_4 u_4,\\
D &= k_1 e_1 + k_2 e_2 + \varepsilon_1 u_1 + 2\beta_2 u_3 - (\beta_1 + \eta_1) u_4,
\end{aligned}
$$

and

$$
\begin{cases}
\beta_1\beta_2 = \beta_1\beta_4 = 0,\\
\beta_2(\beta_1 + \eta_1) = 0,\\
\beta_4(\beta_1 - \eta_1) = 0,\\
\beta_2\beta_4(2\beta_1 - \eta_1) = 0,\\
\beta_2\beta_4(2\beta_1 + \eta_1) = 0,\\
\beta_2(\beta_1 - 2\eta_1) = 0,\\
\beta_4(\beta_1 + 2\eta_1) = 0,\\
4\beta_2\delta_1 + 4\beta_4\varepsilon_1 - \beta_1\eta_1 - k_1 - d_2 = 0,\\
5\beta_2 d_1 + 3\beta_4 k_1 - 2\beta_2\beta_4\delta_1 = 0,\\
3\beta_2 d_2 + 5\beta_4 k_2 - 2\beta_2\beta_4\varepsilon_1 = 0.
\end{cases}
$$

The rest of the proof is similar to that of Proposition 1.4.
This completes the proof of the Proposition.

3. Three-dimensional case

Proposition 3.1. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.1 is equivalent to the trivial pair:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	e_3	u_1	0	$-u_3$	0
e_2	0	0	$-e_3$	0	u_2	0	$-u_4$
e_3	$-e_3$	e_3	0	0	u_1	$-u_4$	0
u_1	$-u_1$	0	0	0	0	0	0
u_2	0	$-u_2$	$-u_1$	0	0	0	0
u_3	u_3	0	u_4	0	0	0	0
u_4	0	u_4	0	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors e_1, e_2.

Lemma. *Any virtual structure q on generalized module 3.1 is equivalent to the trivial.*

Proof. Let q be a virtual structure on generalized module 3.1. By Proposition 6, Chapter I, without loss of generality it can be assumed that q is primary.

Since

$$\begin{aligned} &\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, && \mathfrak{g}^{(1,-1)}(\mathfrak{h}) = \mathbb{C}e_3, \\ &U^{(1,0)}(\mathfrak{h}) = \mathbb{C}u_1, && U^{(0,1)}(\mathfrak{h}) = \mathbb{C}u_2, \\ &U^{(-1,0)}(\mathfrak{h}) = \mathbb{C}u_3, && U^{(0,-1)}(\mathfrak{h}) = \mathbb{C}u_4, \end{aligned}$$

we have

$$C(e_1) = C(e_2) = C(e_3) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Then

$$
\begin{array}{lll}
[e_1, e_2] = 0, & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, \\
[e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, \\
[e_1, u_3] = -u_3, & [e_2, u_3] = 0, & [e_3, u_3] = -u_4, \\
[e_1, u_4] = 0, & [e_2, u_4] = -u_4, & [e_3, u_4] = 0.
\end{array}
$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I).

Thus

$$
\begin{array}{ll}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, & \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{C}e_3, \\
\bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{C}u_1, & \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{C}u_2, \\
\bar{\mathfrak{g}}^{(-1,0)}(\mathfrak{h}) = \mathbb{C}u_3, & \bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) = \mathbb{C}u_4.
\end{array}
$$

Since

$$
\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}), \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}), \\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}), \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-1,1)}(\mathfrak{h}), \\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}), \\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(-1,-1)}(\mathfrak{h}),
\end{aligned}
$$

we have

$$
\begin{aligned}
&[u_1, u_2] = 0, \\
&[u_1, u_3] = ae_1 + be_2, \\
&[u_1, u_4] = ce_3, \\
&[u_2, u_3] = 0, \\
&[u_2, u_4] = de_1 + fe_2, \\
&[u_3, u_4] = 0.
\end{aligned}
$$

Using the Jacobi identity we obtain: $a = b = c = d = f = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

The proof of the Proposition is complete.

Proposition 3.2. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 3.2 is equivalent to one and only one of the following pairs:*

$\lambda = 0$

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2e_2$
e_3	$-e_3$	0	0	0	$-2e_3$	$-u_2$	u_1
u_1	$-u_1$	0	0	0	$2e_3-u_1$	u_2+u_4	$2e_2-u_1$
u_2	0	$-u_1$	$2e_3$	u_1-2e_3	0	$-2u_3$	u_2-u_4
u_3	u_3	u_4	u_2	$-u_2-u_4$	$2u_3$	0	$2u_3$
u_4	0	$2e_2$	$-u_1$	u_1-2e_2	u_4-u_2	$-2u_3$	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2e_2$
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	u_2	$-u_1$
u_2	0	$-u_1$	0	0	0	0	u_2
u_3	u_3	u_4	u_2	$-u_2$	0	0	$2u_3$
u_4	0	$2e_2$	$-u_1$	u_1	$-u_2$	$-2u_3$	0

$\lambda = 1$

3.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	0	u_1	$-u_4$	0
e_3	$-2e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	e_2	0
u_3	u_3	u_4	u_2	0	$-e_2$	0	0
u_4	u_4	0	$-u_1$	0	0	0	0

$\operatorname{Re}\lambda > 0$ *or* $\operatorname{Re}\lambda = 0,\ \operatorname{Im}\lambda \geqslant 0$

4.

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$(1-\lambda)e_2$	$(1+\lambda)e_3$	u_1	λu_2	$-u_3$	$-\lambda u_4$
e_2	$(\lambda-1)e_2$	0	0	0	u_1	$-u_4$	0
e_3	$-(1+\lambda)e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	u_4	u_2	0	0	0	0
u_4	λu_4	0	$-u_1$	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1)=\begin{pmatrix}0&0&0\\0&1-\lambda&0\\0&0&1+\lambda\end{pmatrix},\quad A(e_2)=\begin{pmatrix}0&0&0\\\lambda-1&0&0\\0&0&0\end{pmatrix},$$

$$A(e_3)=\begin{pmatrix}0&0&0\\0&0&0\\-\lambda-1&0&0\end{pmatrix},$$

and for $x\in\mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.2 is equivalent to one of the following:*

a) $\lambda=0$

$$C_1(e_1)=0,\quad C_1(e_2)=\begin{pmatrix}0&0&0&0\\0&0&0&p\\0&0&0&r\end{pmatrix},\quad C_1(e_3)=\begin{pmatrix}0&0&0&0\\0&s&0&0\\0&t&0&0\end{pmatrix};$$

b) $\lambda=\frac{1}{2}$

$$C_2(e_1)=0,\quad C_2(e_2)=\begin{pmatrix}0&0&0&-\frac{2}{3}p\\0&0&0&0\\p&0&0&0\end{pmatrix},\quad C_2(e_3)=0;$$

c) $\lambda=2$

$$C_3(e_1)=\begin{pmatrix}0&0&0&0\\0&0&p&0\\0&0&0&0\end{pmatrix},\quad C_3(e_2)=0,\quad C_3(e_3)=0;$$

d) $\lambda\notin\{0,\frac{1}{2},2\}$

$$C_4(e_i)=0,\quad i=1,2,3.$$

Proof. Let q be a virtual structure on generalized module 3.2. Without loss of generality it can be assumed that q is primary. Consider the following cases:

a)$\lambda=0$. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h})=\mathbb{C}e_1,\quad \mathfrak{g}^{(1)}(\mathfrak{h})=\mathbb{C}e_2\oplus\mathbb{C}e_3,$$

$$U^{(1)}(\mathfrak{h})=\mathbb{C}u_1,\quad U^{(0)}(\mathfrak{h})=\mathbb{C}u_2\oplus\mathbb{C}u_4,\quad U^{(-1)}(\mathfrak{h})=\mathbb{C}u_3,$$

we have

$$C(e_1)=\begin{pmatrix}0&c_3&0&c_4\\c_1&0&0&0\\c_2&0&0&0\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&0&c_7&0\\0&c_5&0&c_8\\0&c_6&0&c_9\end{pmatrix},$$

$$C(e_3)=\begin{pmatrix}0&0&c_{12}&0\\0&c_{10}&0&c_{13}\\0&c_{11}&0&c_{14}\end{pmatrix}.$$

Checking condition (3), Chapter I, for e_i, $i=1,2,3$, we obtain:

$$\begin{cases}c_1=c_2=c_3=c_4=0,\\ c_5=c_{13}-c_{12},\\ c_6=c_7+c_{14}.\end{cases}$$

So, any virtual structure q on generalized module 3.2 has the form:

$$C(e_1)=0,\quad C(e_2)=\begin{pmatrix}0&0&c_7&0\\0&c_{13}-c_{12}&0&c_8\\0&c_7+c_{14}&0&c_9\end{pmatrix},\quad C(e_3)=\begin{pmatrix}0&0&c_{12}&0\\0&c_{10}&0&c_{13}\\0&c_{11}&0&c_{14}\end{pmatrix}.$$

Put

$$H=\begin{pmatrix}0&-c_{12}&0&-c_7\\c_{13}&0&0&0\\c_7+c_{14}&0&0&0\end{pmatrix}.$$

Now put $C_1(x)=C(x)+A(x)H-HB(x)$. Then

$$C_1(e_1)=0,\quad C_1(e_2)=\begin{pmatrix}0&0&0&0\\0&0&0&c_8+c_7\\0&0&0&c_9\end{pmatrix},\quad C_1(e_3)=\begin{pmatrix}0&0&0&0\\0&c_{10}&0&0\\0&c_{11}+c_{12}&0&0\end{pmatrix}.$$

By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent.

b)$\lambda=\frac{1}{2}$. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h})=\mathbb{C}e_1,\quad \mathfrak{g}^{(\frac{1}{2})}(\mathfrak{h})=\mathbb{C}e_2,\quad \mathfrak{g}^{(\frac{3}{2})}(\mathfrak{h})=\mathbb{C}e_3,$$

$$U^{(1)}(\mathfrak{h})=\mathbb{C}u_1,\quad U^{(\frac{1}{2})}(\mathfrak{h})=\mathbb{C}u_2,\quad U^{(-1)}(\mathfrak{h})=\mathbb{C}u_3,\quad U^{(-\frac{1}{2})}(\mathfrak{h})=\mathbb{C}u_4,$$

we have

$$C(e_1)=\begin{pmatrix}0&0&0&0\\0&c_1&0&0\\0&0&0&0\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&0&0&c_3\\0&0&0&0\\c_2&0&0&0\end{pmatrix},$$

$$C(e_3)=\begin{pmatrix}0&0&0&0\\0&0&c_4&0\\0&0&0&0\end{pmatrix}.$$

Checking condition (3), Chapter I, for e_i, $i=1,2,3$, we obtain: $c_1=0$, $c_3=-\frac{2}{3}c_2$. So, any virtual structure q on generalized module 3.2 has the form:

$$C(e_1)=0,\quad C(e_2)=\begin{pmatrix}0&0&0&-\frac{2}{3}c_2\\0&0&0&0\\c_2&0&0&0\end{pmatrix},\quad C(e_3)=\begin{pmatrix}0&0&0&0\\0&0&c_4&0\\0&0&0&0\end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now put $C_2(x) = C(x) + A(x)H - HB(x)$. Then

$$C_2(e_1) = 0, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & 0 & -\frac{2}{3}c_2 \\ 0 & 0 & 0 & 0 \\ c_2 & 0 & 0 & 0 \end{pmatrix}, \quad C_2(e_3) = 0.$$

By Proposition 4, Chapter I, the virtual structures C and C_2 are equivalent.

c)$\lambda = 2$. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \mathfrak{g}^{(-1)}(\mathfrak{h}) = \mathbb{C}e_2, \quad \mathfrak{g}^{(3)}(\mathfrak{h}) = \mathbb{C}e_3,$$

$$U^{(1)}(\mathfrak{h}) = \mathbb{C}u_1, \quad U^{(2)}(\mathfrak{h}) = \mathbb{C}u_2, \quad U^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3, \quad U^{(-2)}(\mathfrak{h}) = \mathbb{C}u_4,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = 0.$$

Checking condition (3), Chapter I, for e_i, $i = 1, 2, 3$, we obtain: $c_2 = 0$, and the virtual structure q on generalized module 3.2 has the form:

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = C(e_3) = 0.$$

Now put $C_3 = C$.

d)$\lambda \notin \{0, \frac{1}{2}, 2\}$. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) \supset \mathbb{C}e_1, \quad \mathfrak{g}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{C}e_2, \quad \mathfrak{g}^{(1+\lambda)}(\mathfrak{h}) \supset \mathbb{C}e_3,$$

$$U^{(1)}(\mathfrak{h}) \supset \mathbb{C}u_1, \quad U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{C}u_2, \quad U^{(-1)}(\mathfrak{h}) \supset \mathbb{C}u_3, \quad U^{(-\lambda)}(\mathfrak{h}) \supset \mathbb{C}u_4,$$

we have

$$C(e_1) = C(e_2) = C(e_3) = 0.$$

So, the virtual structure q on generalized module 3.2 is equivalent to the trivial. Put $C_4 = C$.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Consider the following cases:

1°. $\lambda = 0$. Then

$$\begin{array}{lll}
[e_1, e_2] = e_2, & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, \\
[e_1, u_2] = 0, & [e_2, u_2] = u_1, & [e_3, u_2] = se_2 + te_3, \\
[e_1, u_3] = -u_3, & [e_2, u_3] = -u_4, & [e_3, u_3] = -u_2, \\
[e_1, u_4] = 0, & [e_2, u_4] = pe_2 + re_3, & [e_3, u_4] = u_1.
\end{array}$$

Since

$$\begin{aligned}
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) &= \mathbb{C}e_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_4, \\
\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) &= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}u_1, \\
\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) &= \mathbb{C}u_3,
\end{aligned}$$

and

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}), \\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),
\end{aligned}$$

we have:

$$\begin{aligned}
[u_1, u_2] &= ae_2 + be_3 + \alpha u_1, \\
[u_1, u_3] &= ce_1 + \beta_1 u_2 + \beta_2 u_4, \\
[u_1, u_4] &= de_2 + fe_3 + \gamma u_1, \\
[u_2, u_3] &= \delta u_3, \\
[u_2, u_4] &= ke_1 + \eta_1 u_2 + \eta_2 u_4, \\
[u_3, u_4] &= \varepsilon u_3.
\end{aligned}$$

Using the Jacobi identity we obtain:

$$\begin{cases}
a = c = f = 0, \\
k = r = s = 0, \\
\eta_1 = -\gamma = \beta_1, \\
\eta_2 = \alpha = -\beta_2, \\
\delta = t = -2\beta_2, \\
\varepsilon = -p = 2\beta_1, \\
b = d = 2\beta_1\beta_2.
\end{cases}$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	e_2	e_3	u_1	0	$-u_3$	0
e_2	$-e_2$	0	0	0	u_1	$-u_4$	$-2\beta_1 e_2$
e_3	$-e_3$	0	0	0	$-2\beta_2 e_3$	$-u_2$	u_1
u_1	$-u_1$	0	0	0	$2\beta_1\beta_2 e_3-\beta_2 u_1$	$\beta_1 u_2+\beta_2 u_4$	$2\beta_1\beta_2 e_2-\beta_1 u_1$
u_2	0	$-u_1$	$2\beta_2 e_3$	$\beta_2 u_1-2\beta_1\beta_2 e_3$	0	$-2\beta_2 u_3$	$\beta_1 u_2-\beta_2 u_4$
u_3	u_3	u_4	u_2	$-\beta_1 u_2-\beta_2 u_4$	$2\beta_2 u_3$	0	$2\beta_1 u_3$
u_4	0	$2\beta_1 e_2$	$-u_1$	$\beta_1 u_1-2\beta_1\beta_2 e_2$	$\beta_2 u_4-\beta_1 u_2$	$-2\beta_1 u_3$	0

Consider the following cases:

1.1°. $\beta_1\beta_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(e_3) &= \frac{\beta_1}{\beta_2} e_3,\\
\pi(u_1) &= \frac{1}{\beta_2} u_1,\\
\pi(u_2) &= \frac{1}{\beta_2} u_2,\\
\pi(u_3) &= \frac{1}{\beta_1} u_3,\\
\pi(u_4) &= \frac{1}{\beta_1} u_4.
\end{aligned}$$

1.2°. $\beta_1 \neq 0,\ \beta_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(e_3) &= e_3,\\
\pi(u_1) &= \frac{1}{\beta_1} u_1,\\
\pi(u_2) &= \frac{1}{\beta_1} u_2,\\
\pi(u_3) &= \frac{1}{\beta_1} u_3,\\
\pi(u_4) &= \frac{1}{\beta_1} u_4.
\end{aligned}$$

1.3°. $\beta_1 = 0,\ \beta_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_3,\\
\pi(e_3) &= e_2,\\
\pi(u_1) &= \frac{1}{\beta_2}u_1,\\
\pi(u_2) &= \frac{1}{\beta_2}u_4,\\
\pi(u_3) &= \frac{1}{\beta_2}u_3,\\
\pi(u_4) &= \frac{1}{\beta_2}u_2.
\end{aligned}$$

1.4°. $\beta_1 = \beta_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 = 6$, $\dim \mathcal{D}^2\bar{\mathfrak{g}}_2 = 5$, $\dim \mathcal{D}^2\bar{\mathfrak{g}}_4 = 3$ we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ are not equivalent to each other.

2°. $\lambda = \frac{1}{2}$. Then

$$\begin{array}{lll}
[e_1, e_2] = \frac{1}{2}e_2, & & \\
[e_1, e_3] = \frac{3}{2}e_3, & [e_2, e_3] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = pe_3, & [e_3, u_1] = 0,\\
[e_1, u_2] = \frac{1}{2}u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0,\\
[e_1, u_3] = -u_3, & [e_2, u_3] = -u_4, & [e_3, u_3] = -u_2,\\
[e_1, u_4] = -\frac{1}{2}u_4, & [e_2, u_4] = -\frac{2}{3}pe_1, & [e_3, u_4] = u_1.
\end{array}$$

Since

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, & \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1,\\
\bar{\mathfrak{g}}^{(\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}e_2 \oplus \mathbb{C}u_2, & \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3,\\
\bar{\mathfrak{g}}^{(\frac{3}{2})}(\mathfrak{h}) = \mathbb{C}e_3, & \bar{\mathfrak{g}}^{(-\frac{1}{2})}(\mathfrak{h}) = \mathbb{C}u_4,
\end{array}$$

and

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(\frac{3}{2})}(\mathfrak{h}),\\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(\frac{1}{2})}(\mathfrak{h}),\\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(-\frac{1}{2})}(\mathfrak{h}),\\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(-\frac{3}{2})}(\mathfrak{h}),
\end{aligned}$$

we have:

$$\begin{aligned}
[u_1, u_2] &= ae_3,\\
[u_1, u_3] &= be_1,\\
[u_1, u_4] &= ce_2 + \alpha u_2,\\
[u_2, u_3] &= \beta u_4,\\
[u_2, u_4] &= de_1,\\
[u_3, u_4] &= 0.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = d = p = \alpha = \beta = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_4,\mathfrak{g}_4)$.

3°. $\lambda = 2$. Then

$$\begin{array}{lll}
[e_1, e_2] = -e_2, & & \\
[e_1, e_3] = 3e_3, & [e_2, e_3] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0,\\
[e_1, u_2] = 2u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0,\\
[e_1, u_3] = pe_2 - u_3, & [e_2, u_3] = -u_4, & [e_3, u_3] = -u_2,\\
[e_1, u_4] = -2u_4, & [e_2, u_4] = 0, & [e_3, u_4] = u_1.
\end{array}$$

Since

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, & \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1,\\
\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}e_2 \oplus \mathbb{C}u_3, & \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{C}u_2,\\
\bar{\mathfrak{g}}^{(3)}(\mathfrak{h}) = \mathbb{C}e_3, & \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) = \mathbb{C}u_4,
\end{array}$$

and

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(3)}(\mathfrak{h}),\\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),\\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),\\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(-3)}(\mathfrak{h}),
\end{aligned}$$

we have:

$$\begin{aligned}
[u_1, u_2] &= ae_3,\\
[u_1, u_3] &= be_1,\\
[u_1, u_4] &= ce_2 + \alpha u_3,\\
[u_2, u_3] &= \beta u_1,\\
[u_2, u_4] &= de_1,\\
[u_3, u_4] &= 0.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = d = p = \alpha = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	$3e_3$	u_1	$2u_2$	$-u_3$	$-2u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-3e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-2u_2$	$-u_1$	0	0	0	βu_1	0
u_3	u_3	u_4	u_2	0	$-\beta u_1$	0	0
u_4	$2u_4$	0	$-u_1$	0	0	0	0 .

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(e_3) &= e_3, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2, \\
\pi(u_3) &= \beta e_2 + u_3, \\
\pi(u_4) &= u_4.
\end{aligned}$$

4°. $\lambda = 1$. Then

$$\begin{array}{lll}
[e_1, e_2] = 0, & & \\
[e_1, e_3] = 2e_3, & [e_2, e_3] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, \\
[e_1, u_2] = u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0, \\
[e_1, u_3] = -u_3, & [e_2, u_3] = -u_4, & [e_3, u_3] = -u_2, \\
[e_1, u_4] = -u_4, & [e_2, u_4] = 0, & [e_3, u_4] = u_1.
\end{array}$$

Since

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, & \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_2, \\
\bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{C}e_3, & \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_3 \oplus \mathbb{C}u_4,
\end{array}$$

and

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), \\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}),
\end{aligned}$$

we have:

$$
\begin{aligned}
[u_1, u_2] &= ae_3,\\
[u_1, u_3] &= be_1 + ce_2,\\
[u_1, u_4] &= de_1 + fe_2,\\
[u_2, u_3] &= me_1 + ne_2,\\
[u_2, u_4] &= ke_1 + re_2,\\
[u_3, u_4] &= 0.
\end{aligned}
$$

Using the Jacobi identity we obtain: $a = b = c = d = f = m = k = r = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	u_1	u_2	$-u_3$	$-u_4$
e_2	0	0	0	0	u_1	$-u_4$	0
e_3	$-2e_3$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	ne_2	0
u_3	u_3	u_4	u_2	0	$-ne_2$	0	0
u_4	u_4	0	$-u_1$	0	0	0	0 .

Consider the following cases:

4.1°. $n \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$
\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= ne_2,\\
\pi(e_3) &= e_3,\\
\pi(u_1) &= nu_1,\\
\pi(u_2) &= u_2,\\
\pi(u_3) &= u_3,\\
\pi(u_4) &= nu_4.
\end{aligned}
$$

4.2°. $n = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_3 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_4$ we see that the pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ are not equivalent.

5°. $\lambda \notin \{0, \frac{1}{2}, 1, 2\}$. Then

$$
\begin{array}{lll}
[e_1, e_2] = (1 - \lambda)e_2, & & \\
[e_1, e_3] = (1 + \lambda)e_3, & [e_2, e_3] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0,\\
[e_1, u_2] = \lambda u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0,\\
[e_1, u_3] = -u_3, & [e_2, u_3] = -u_4, & [e_3, u_3] = -u_2,\\
[e_1, u_4] = -\lambda u_4, & [e_2, u_4] = -0, & [e_3, u_4] = u_1.
\end{array}
$$

Since

$$\begin{aligned}
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) &= \mathbb{C}e_1,\\
\bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}) &= \mathbb{C}e_2,\\
\bar{\mathfrak{g}}^{(1+\lambda)}(\mathfrak{h}) &= \mathbb{C}e_3,\\
\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) &= \mathbb{C}u_1,\\
\bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}) &= \mathbb{C}u_2,\\
\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) &= \mathbb{C}u_3,\\
\bar{\mathfrak{g}}^{(-\lambda)}(\mathfrak{h}) &= \mathbb{C}u_4,
\end{aligned}$$

and

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(1+\lambda)}(\mathfrak{h}),\\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}),\\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(\lambda-1)}(\mathfrak{h}),\\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(-1-\lambda)}(\mathfrak{h}),
\end{aligned}$$

we have:

$$\begin{aligned}
[u_1, u_2] &= ae_3,\\
[u_1, u_3] &= be_1,\\
[u_1, u_4] &= ce_2,\\
[u_2, u_3] &= 0,\\
[u_2, u_4] &= de_1,\\
[u_3, u_4] &= 0.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = d = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$.

This completes the proof of the Proposition.

Proposition 3.3. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.3 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	u_1	0
u_2	$-u_2$	$-u_1$	0	0	0	pe_3+u_2	0
u_3	0	u_4	u_2	$-u_1$	$-pe_3-u_2$	0	$-pe_2-u_4$
u_4	u_4	0	$-u_1$	0	0	pe_2+u_4	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	e_3	0
u_3	0	u_4	u_2	0	$-e_3$	0	$-e_2$
u_4	u_4	0	$-u_1$	0	0	e_2	0

3.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0	0
u_3	0	u_4	u_2	0	0	0	0
u_4	u_4	0	$-u_1$	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.3 is equivalent to the following:*

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & p & 0 \\ 0 & 0 & 0 & -p \\ 0 & p & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = 0, \quad C_1(e_3) = 0.$$

Proof. Let q be a virtual structure on generalized module 3.3. Without loss of generality it can be assumed that q is primary.

Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \qquad U^{(1)}(\mathfrak{h}) = \mathbb{C}u_2,$$
$$\mathfrak{g}^{(-1)}(\mathfrak{h}) = \mathbb{C}e_2, \qquad U^{(0)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_3,$$
$$\mathfrak{g}^{(1)}(\mathfrak{h}) = \mathbb{C}e_3, \qquad U^{(-1)}(\mathfrak{h}) = \mathbb{C}u_4,$$

we have

$$C(e_1)=\begin{pmatrix} c_1 & 0 & c_3 & 0\\ 0 & 0 & 0 & c_4\\ 0 & c_2 & 0 & 0\end{pmatrix},\quad C(e_2)=\begin{pmatrix} 0 & c_6 & 0 & 0\\ c_5 & 0 & c_7 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$C(e_3)=\begin{pmatrix} 0 & 0 & 0 & c_{10}\\ 0 & 0 & 0 & 0\\ c_8 & 0 & c_9 & 0\end{pmatrix}.$$

Checking condition (3), Chapter I, for e_i, $i=1,2,3$, we obtain:

$$\begin{cases} c_1=0,\\ c_2=c_3=-c_4,\\ c_6=c_8=c_{10}=-c_5.\end{cases}$$

So, any virtual structure q on generalized module 3.3 has the form:

$$C(e_1)=\begin{pmatrix} 0 & 0 & c_2 & 0\\ 0 & 0 & 0 & -c_2\\ 0 & c_2 & 0 & 0\end{pmatrix},\quad C(e_2)=\begin{pmatrix} 0 & c_6 & 0 & 0\\ -c_6 & 0 & c_7 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$C(e_3)=\begin{pmatrix} 0 & 0 & 0 & c_6\\ 0 & 0 & 0 & 0\\ c_6 & 0 & c_9 & 0\end{pmatrix}.$$

Put

$$H=\begin{pmatrix} c_6 & 0 & 0 & 0\\ 0 & 0 & 0 & -c_7\\ 0 & -c_9 & 0 & 0\end{pmatrix}.$$

Now put $C_1(x)=C(x)+A(x)H-HB(x)$. Then

$$C_1(e_1)=\begin{pmatrix} 0 & 0 & c_2 & 0\\ 0 & 0 & 0 & -c_2\\ 0 & c_2 & 0 & 0\end{pmatrix},\quad C_1(e_2)=C_1(e_3)=0.$$

By Proposition 4, Chapter I, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of type 3.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is defined by the virtual structure determined in the Lemma.

Then

$$\begin{array}{lll}
[e_1,e_2]=-e_2, & & \\
[e_1,e_3]=e_3, & [e_2,e_3]=0, & \\
[e_1,u_1]=0, & [e_2,u_1]=0, & [e_3,u_1]=0,\\
[e_1,u_2]=pe_3+u_2, & [e_2,u_2]=u_1, & [e_3,u_2]=0,\\
[e_1,u_3]=pe_1, & [e_2,u_3]=-u_4, & [e_3,u_3]=-u_2,\\
[e_1,u_4]=-pe_2-u_4, & [e_2,u_4]=0, & [e_3,u_4]=u_1.
\end{array}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\begin{aligned}
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) &= \mathbb{C}e_1 \oplus \mathbb{C}u_1 \oplus \mathbb{C}u_3,\\
\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) &= \mathbb{C}e_2 \oplus \mathbb{C}u_4,\\
\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) &= \mathbb{C}e_3 \oplus \mathbb{C}u_2.
\end{aligned}$$

Since

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),\\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),\\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),\\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),
\end{aligned}$$

we have:

$$\begin{aligned}
[u_1, u_2] &= ae_3 + \alpha u_2,\\
[u_1, u_3] &= be_1 + \beta_1 u_1 + \beta_2 u_3,\\
[u_1, u_4] &= ce_2 + \gamma u_4,\\
[u_2, u_3] &= de_3 + \delta u_2,\\
[u_2, u_4] &= fe_1 + \eta_1 u_1 + \eta_2 u_3,\\
[u_3, u_4] &= ke_2 + \varepsilon u_4.
\end{aligned}$$

Using the Jacobi identity we obtain:

$$\begin{cases}
a = b = c = 0,\\
f = p = \alpha = 0,\\
\beta_2 = \eta_2 = \gamma = 0,\\
\delta = \beta_1 - \eta_1,\\
\varepsilon = -\beta_1 - \eta_1,\\
k = -d - \beta_1\eta_1.
\end{cases}$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	$\beta_1 u_1$	0
u_2	$-u_2$	$-u_1$	0	0	0	$de_3+(\beta_1-\eta_1)u_2$	$\eta_1 u_1$
u_3	0	u_4	u_2	$-\beta_1 u_1$	$-de_3-(\beta_1-\eta_1)u_2$	0	A
u_4	u_4	0	$-u_1$	0	$-\eta_1 u_1$	$-A$	0 ,

where $A = -(d+\beta_1\eta_1)e_2-(\beta_1+\eta_1)u_4$.

The pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(e_3) &= e_3,\\
\pi(u_1) &= u_1,\\
\pi(u_2) &= -\frac{\eta_1}{2}e_3 + u_2,\\
\pi(u_3) &= -\frac{\eta_1}{2}e_1 + u_3,\\
\pi(u_4) &= \frac{\eta_1}{2}e_2 + u_4.
\end{aligned}$$

The pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ has the form:

[,]	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$-e_2$	e_3	0	u_2	0	$-u_4$
e_2	e_2	0	0	0	u_1	$-u_4$	0
e_3	$-e_3$	0	0	0	0	$-u_2$	u_1
u_1	0	0	0	0	0	$\beta_1 u_1$	0
u_2	$-u_2$	$-u_1$	0	0	0	$de_3+\beta_1 u_2$	0
u_3	0	u_4	u_2	$-\beta_1 u_1$	$-de_3-\beta_1 u_2$	0	$-de_2-\beta_1 u_4$
u_4	u_4	0	$-u_1$	0	0	$de_2+\beta_1 u_4$	0

Consider the following cases:

1°. $\beta_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(e_3) &= e_3,\\
\pi(u_1) &= \frac{1}{\beta_1}u_1,\\
\pi(u_2) &= \frac{1}{\beta_1}u_2,\\
\pi(u_3) &= \frac{1}{\beta_1}u_3,\\
\pi(u_4) &= \frac{1}{\beta_1}u_4.
\end{aligned}$$

2°. $\beta_1 = 0,\ d \neq 0$. Then the pair $(\bar{\mathfrak{g}}',\mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(e_3) &= e_3,\\
\pi(u_1) &= \frac{1}{\sqrt{d}}u_1,\\
\pi(u_2) &= \frac{1}{\sqrt{d}}u_2,\\
\pi(u_3) &= \frac{1}{\sqrt{d}}u_3,\\
\pi(u_4) &= \frac{1}{\sqrt{d}}u_4.
\end{aligned}$$

3°. $\beta_1 = d = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$.

It remains to show that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, 2, 3$, are not equivalent to each other.

Consider the algebras

$$\tilde{\mathfrak{g}}_i = \bar{\mathfrak{g}}_i / \mathcal{D}^2 \bar{\mathfrak{g}}_i, i = 1, 2, 3$$

and the homomorphisms

$$f_i : \tilde{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{C}) \ (i = 1, 2, 3),$$

where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{\mathcal{D}\tilde{\mathfrak{g}}_i} x$ in the basis $\{e_2 + \mathbb{C}u_1, u_4 + \mathbb{C}u_1, e_3 + \mathbb{C}u_1, u_2 + \mathbb{C}u_1\}$, $x \in \tilde{\mathfrak{g}}_i$. We have:

$$f_1(\tilde{\mathfrak{g}}_1) = \left\{ \left. \begin{pmatrix} -x & -py & 0 & 0 \\ y & -x-y & 0 & 0 \\ 0 & 0 & x & -py \\ 0 & 0 & y & x-y \end{pmatrix} \right| x, y \in \mathbb{C} \right\},$$

$$f_2(\tilde{\mathfrak{g}}_2) = \left\{ \left. \begin{pmatrix} -x & -y & 0 & 0 \\ y & -x & 0 & 0 \\ 0 & 0 & x & -y \\ 0 & 0 & y & x \end{pmatrix} \right| x, y \in \mathbb{C} \right\},$$

$$f_3(\tilde{\mathfrak{g}}_3) = \left\{ \left. \begin{pmatrix} -x & 0 & 0 & 0 \\ y & -x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & y & x \end{pmatrix} \right| x, y \in \mathbb{C} \right\}.$$

Since the subalgebras $f_1(\tilde{\mathfrak{g}}_1)$, $f_2(\tilde{\mathfrak{g}}_2)$, and $f_3(\tilde{\mathfrak{g}}_3)$ of the Lie algebra $\mathfrak{gl}(4, \mathbb{C})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent to each other.

Consider the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_1', \mathfrak{g}_1')$ with parameters p and p' respectively.

Since the subalgebras $f_1(\tilde{\mathfrak{g}}_1)$ and ${f_1}'(\tilde{\mathfrak{g}}_1')$ of the Lie algebra $\mathfrak{gl}(4, \mathbb{C})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_1', \mathfrak{g}_1')$ are not equivalent, whenever $p \neq p'$.

This completes the proof of the Proposition.

Proposition 3.4. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type* 3.4 *is equivalent to the trivial pair:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0
u_3	u_3	u_4	0	0	0	0	0
u_4	$-u_4$	0	u_3	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.4 is equivalent to the trivial.*

Proof. Note that $\mathfrak{g}$ is a semisimple Lie algebra. By Proposition 9, Chapter I, without loss of generality it can be assumed that $q(\mathfrak{g}) = \{0\}$. This completes the proof of the Lemma.

Let $\bar{\mathfrak{g}}$ be a pair of type 3.4. Then it can be assumed that the corresponding virtual pair $\bar{\mathfrak{g}}$ is defined by the trivial virtual structure.

Then

$$\begin{array}{lll} [e_1, e_2] = 2e_2, & & \\ [e_1, e_3] = -2e_3, & [e_2, e_3] = e_1, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = u_2, \\ [e_1, u_2] = -u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0, \\ [e_1, u_3] = -u_3, & [e_2, u_3] = -u_4, & [e_3, u_3] = 0, \\ [e_1, u_4] = u_4, & [e_2, u_4] = 0, & [e_3, u_4] = -u_3. \end{array}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1, \quad \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{C}e_2, \quad \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) = \mathbb{C}e_3,$$
$$\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_4, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_2 \oplus \mathbb{C}u_3.$$

Since

$$\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}),\\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}),\\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),
\end{aligned}$$

we have

$$\begin{aligned}
[u_1, u_2] &= ae_1,\\
[u_1, u_3] &= be_1,\\
[u_1, u_4] &= ce_2,\\
[u_2, u_3] &= de_3,\\
[u_2, u_4] &= fe_1,\\
[u_3, u_4] &= ke_1.
\end{aligned}$$

Using the Jacobi identity we obtain: $a = b = c = d = f = k = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

This completes the proof of the Proposition.

Proposition 3.5. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.5 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	u_1
u_2	0	$-u_1$	u_3	0	0	0	u_2
u_3	$2u_3$	$2u_2$	0	0	0	0	u_3
u_4	0	0	0	$-u_1$	$-u_2$	$-u_3$	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	e_2	e_1	0
u_2	0	$-u_1$	u_3	$-e_2$	0	e_3	0
u_3	$2u_3$	$2u_2$	0	$-e_1$	$-e_3$	0	0
u_4	0	0	0	0	0	0	0

3.

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	0	0	0
u_2	0	$-u_1$	u_3	0	0	0	0
u_3	$2u_3$	$2u_2$	0	0	0	0	0
u_4	0	0	0	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.5 is equivalent to the trivial.*

Proof. Note that $\mathfrak{g}$ is a semisimple Lie algebra. By Proposition 9, Chapter I, without loss of generality it can be assumed that $q(\mathfrak{g}) = \{0\}$. This completes the proof of the Lemma.

Let $\bar{\mathfrak{g}}$ be a pair of type 3.5. Then it can be assumed that the corresponding virtual pair $\bar{\mathfrak{g}}$ is defined by the trivial virtual structure.

Then

$$\begin{aligned}
&[e_1, e_2] = 2e_2, && && \\
&[e_1, e_3] = -2e_3, && [e_2, e_3] = e_1, && \\
&[e_1, u_1] = 2u_1, && [e_2, u_1] = 0, && [e_3, u_1] = 2u_2, \\
&[e_1, u_2] = 0, && [e_2, u_2] = u_1, && [e_3, u_2] = -u_3, \\
&[e_1, u_3] = -2u_3, && [e_2, u_3] = -2u_2, && [e_3, u_3] = 0, \\
&[e_1, u_4] = 0, && [e_2, u_4] = 0, && [e_3, u_4] = 0.
\end{aligned}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}u_4, \quad \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{C}e_2 \oplus \mathbb{C}u_1, \quad \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) = \mathbb{C}e_3 \oplus \mathbb{C}u_3.$$

Since

$$
\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}),\\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}),\\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}),\\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),\\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}),
\end{aligned}
$$

we have

$$
\begin{aligned}
&[u_1, u_2] = ae_2 + \alpha u_1,\\
&[u_1, u_3] = be_1 + \beta_1 u_2 + \beta_2 u_4,\\
&[u_1, u_4] = ce_2 + \gamma u_1,\\
&[u_2, u_3] = de_3 + \delta u_3,\\
&[u_2, u_4] = fe_1 + \eta_1 u_2 + \eta_2 u_4,\\
&[u_3, u_4] = ke_3 + \varepsilon u_3.
\end{aligned}
$$

Using the Jacobi identity we obtain:

$$
\begin{cases}
\beta_2 = \eta_2 = 0,\\
a = b = d,\\
\delta = \alpha,\\
\beta_1 = -2\alpha,\\
\eta_1 = \varepsilon = \gamma,\\
c = k = -\frac{\alpha\gamma}{2},\\
f = \frac{\alpha\gamma}{4},\\
a\gamma + \frac{\alpha^2\gamma}{4} = 0.
\end{cases}
$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	$ae_2 + \alpha u_1$	$ae_1 - 2\alpha u_2$	$-\frac{\alpha\gamma}{2}e_2 + \gamma u_1$
u_2	0	$-u_1$	u_3	$-ae_2 - \alpha u_1$	0	$ae_3 + \alpha u_3$	$\frac{\alpha\gamma}{4}e_1 + \gamma u_2$
u_3	$2u_3$	$2u_2$	0	$-ae_1 + 2\alpha u_2$	$-ae_3 - \alpha u_3$	0	$-\frac{\alpha\gamma}{2}e_3 + \gamma u_3$
u_4	0	0	0	$\frac{\alpha\gamma}{2}e_2 - \gamma u_1$	$-\frac{\alpha\gamma}{4}e_1 - \gamma u_2$	$\frac{\alpha\gamma}{2}e_3 - \gamma u_3$	0

,

where $a\gamma + \frac{\alpha^2\gamma}{4} = 0$.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(e_3) &= e_3, \\
\pi(u_1) &= -\frac{\alpha}{2} e_2 + u_1, \\
\pi(u_2) &= \frac{\alpha}{4} e_1 + u_2, \\
\pi(u_3) &= -\frac{\alpha}{2} e_3 + u_3, \\
\pi(u_4) &= u_4.
\end{aligned}$$

The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	$2u_1$	0	$-2u_3$	0
e_2	$-2e_2$	0	e_1	0	u_1	$-2u_2$	0
e_3	$2e_3$	$-e_1$	0	$2u_2$	$-u_3$	0	0
u_1	$-2u_1$	0	$-2u_2$	0	ae_2	ae_1	γu_1
u_2	0	$-u_1$	u_3	$-ae_2$	0	ae_3	γu_2
u_3	$2u_3$	$2u_2$	0	$-ae_1$	$-ae_3$	0	γu_3
u_4	0	0	0	$-\gamma u_1$	$-\gamma u_2$	$-\gamma u_3$	0 ,

where $a\gamma = 0$.

Consider the following cases:

1°. $\gamma \neq 0$. Then $a = 0$ and the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
\pi(e_1) &= e_1, \\
\pi(e_2) &= e_2, \\
\pi(e_3) &= e_3, \\
\pi(u_1) &= \frac{1}{\gamma} u_1, \\
\pi(u_2) &= \frac{1}{\gamma} u_2, \\
\pi(u_3) &= \frac{1}{\gamma} u_3, \\
\pi(u_4) &= \frac{1}{\gamma} u_4.
\end{aligned}$$

2°. $a \neq 0$. Then $\gamma = 0$ and the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}\pi(e_1) &= e_1,\\ \pi(e_2) &= e_2,\\ \pi(e_3) &= e_3,\\ \pi(u_1) &= \frac{1}{\sqrt{a}}u_1,\\ \pi(u_2) &= \frac{1}{\sqrt{a}}u_2,\\ \pi(u_3) &= \frac{1}{\sqrt{a}}u_3,\\ \pi(u_4) &= \frac{1}{\sqrt{a}}u_4.\end{aligned}$$

3°. $a = \gamma = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$.

Since $\mathcal{Z}(\bar{\mathfrak{g}}_1) \neq \{0\}$, $\mathcal{Z}(\bar{\mathfrak{g}}_i) = \{0\}$, $i = 1, 2$ we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, 2$ are not equivalent.

Since $\dim \mathfrak{r}(\bar{\mathfrak{g}}_2) \neq \dim \mathfrak{r}(\bar{\mathfrak{g}}_3)$ we see that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

This completes the proof of the Proposition.

4. Pairs with subalgebras of dimension higher than three

Proposition 4.1. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.1 is equivalent to the trivial pair:*

1.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	e_3	e_4	u_1	0	$-u_3$	0
e_2	0	0	$-e_3$	e_4	0	u_2	0	$-u_4$
e_3	$-e_3$	e_3	0	0	0	u_1	$-u_4$	0
e_4	$-e_4$	$-e_4$	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	0	0	0	0	0	0
u_2	0	$-u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	0	u_4	u_2	0	0	0	0
u_4	0	u_4	0	$-u_1$	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors e_1, e_2.

Lemma. *Any virtual structure q on generalized module 4.1 is equivalent to the trivial.*

Proof. Let q be a virtual structure on generalized module 4.1. By Proposition 6, Chapter I, without loss of generality it can be assumed that q is primary.

Since

$$\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \quad \mathfrak{g}^{(1,-1)}(\mathfrak{h}) = \mathbb{C}e_3, \quad \mathfrak{g}^{(1,1)}(\mathfrak{h}) = \mathbb{C}e_4,$$

$$U^{(1,0)}(\mathfrak{h}) = \mathbb{C}u_1, \quad U^{(0,1)}(\mathfrak{h}) = \mathbb{C}u_2, \quad U^{(-1,0)}(\mathfrak{h}) = \mathbb{C}u_3, \quad U^{(0,-1)}(\mathfrak{h}) = \mathbb{C}u_4,$$

we have
$$C(e_1) = C(e_2) = C(e_3) = C(e_4) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Then
$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & \\
[e_1, e_4] = e_4, & [e_2, e_4] = e_4, & [e_3, e_4] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\
[e_1, u_3] = -u_3, & [e_2, u_3] = 0, & [e_3, u_3] = -u_4, & [e_4, u_3] = -u_2, \\
[e_1, u_4] = 0, & [e_2, u_4] = -u_4, & [e_3, u_4] = 0, & [e_4, u_4] = u_1.
\end{array}$$
Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have
$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$
(Proposition 7, Chapter I).

Thus
$$\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \quad \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{C}e_3, \quad \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}) = \mathbb{C}e_4,$$
$$\bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{C}u_1, \quad \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{C}u_2, \quad \bar{\mathfrak{g}}^{(-1,0)}(\mathfrak{h}) = \mathbb{C}u_3, \quad \bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) = \mathbb{C}u_4.$$
Since
$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}),\\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}),\\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}),\\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-1,1)}(\mathfrak{h}),\\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}),\\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(-1,-1)}(\mathfrak{h}),
\end{aligned}$$
we have
$$\begin{aligned}
&[u_1, u_2] = ae_4,\\
&[u_1, u_3] = be_1 + ce_2,\\
&[u_1, u_4] = de_3,\\
&[u_2, u_3] = 0,\\
&[u_2, u_4] = fe_1 + ke_2,\\
&[u_3, u_4] = 0.
\end{aligned}$$
Using the Jacobi identity we obtain: $a = b = c = d = f = k = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

The proof of the Proposition is complete.

Proposition 4.2. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 4.2 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	e_1+3e_2	$2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	0	$2e_4$	$-e_1+3e_2$
u_3	u_3	u_3	u_4	0	$-e_1-3e_2$	$-2e_4$	0	0
u_4	$-u_4$	u_4	0	u_3	$-2e_3$	e_1-3e_2	0	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	0	0	0
u_3	u_3	u_3	u_4	0	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors e_1, e_2.

Lemma. *Any virtual structure q on generalized module 4.2 is equivalent to the trivial.*

Proof. Let q be a virtual structure on generalized module 4.2. By Proposition 6, Chapter I, without loss of generality it can be assumed that q is primary.

Since

$$\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \quad \mathfrak{g}^{(2,0)}(\mathfrak{h}) = \mathbb{C}e_3, \quad \mathfrak{g}^{(-2,0)}(\mathfrak{h}) = \mathbb{C}e_4,$$
$$U^{(1,1)}(\mathfrak{h}) = \mathbb{C}u_1, \qquad U^{(-1,1)}(\mathfrak{h}) = \mathbb{C}u_2,$$
$$U^{(-1,-1)}(\mathfrak{h}) = \mathbb{C}u_3, \quad U^{(1,-1)}(\mathfrak{h}) = \mathbb{C}u_4,$$

we have

$$C(e_1) = C(e_2) = C(e_3) = C(e_4) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = 2e_3, & [e_2, e_3] = 0, & & \\
[e_1, e_4] = -2e_4, & [e_2, e_4] = 0, & [e_3, e_4] = e_1, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = u_1, & [e_3, u_1] = 0, & [e_4, u_1] = u_2, \\
[e_1, u_2] = -u_2, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\
[e_1, u_3] = -u_3, & [e_2, u_3] = -u_3, & [e_3, u_3] = -u_4, & [e_4, u_3] = 0, \\
[e_1, u_4] = u_4, & [e_2, u_4] = -u_4, & [e_3, u_4] = 0, & [e_4, u_4] = -u_3.
\end{array}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I).

Thus

$$\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \quad \bar{\mathfrak{g}}^{(2,0)}(\mathfrak{h}) = \mathbb{C}e_3, \quad \bar{\mathfrak{g}}^{(-2,0)}(\mathfrak{h}) = \mathbb{C}e_4,$$
$$\bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}) = \mathbb{C}u_1, \qquad \bar{\mathfrak{g}}^{(-1,1)}(\mathfrak{h}) = \mathbb{C}u_2,$$
$$\bar{\mathfrak{g}}^{(-1,-1)}(\mathfrak{h}) = \mathbb{C}u_3, \quad \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{C}u_4.$$

Since

$$[u_1, u_2] \in \bar{\mathfrak{g}}^{(0,2)}(\mathfrak{h}),$$
$$[u_1, u_3] \in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}),$$
$$[u_1, u_4] \in \bar{\mathfrak{g}}^{(2,0)}(\mathfrak{h}),$$
$$[u_2, u_3] \in \bar{\mathfrak{g}}^{(-2,0)}(\mathfrak{h}),$$
$$[u_2, u_4] \in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}),$$
$$[u_3, u_4] \in \bar{\mathfrak{g}}^{(0,-2)}(\mathfrak{h}),$$

we have

$$\begin{aligned}
[u_1, u_2] &= 0,\\
[u_1, u_3] &= ae_1 + be_2,\\
[u_1, u_4] &= ce_3,\\
[u_2, u_3] &= ke_4,\\
[u_2, u_4] &= de_1 + fe_2,\\
[u_3, u_4] &= 0.
\end{aligned}$$

Using the Jacobi identity we obtain:

$$\begin{cases} c = k = 2a,\\ b = f = 3a,\\ d = -a. \end{cases}$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	u_2	0	0	$-u_3$
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	ae_1+3ae_2	$2ae_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	0	$2ae_4$	$-ae_1+3ae_2$
u_3	u_3	u_3	u_4	0	$-ae_1-3ae_2$	$-2ae_4$	0	0
u_4	$-u_4$	u_4	0	u_3	$-2ae_3$	ae_1-3ae_2	0	0 .

Consider the following cases:

1°. $a \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(e_3) &= e_3,\\
\pi(e_4) &= e_4,\\
\pi(u_1) &= \frac{1}{\sqrt{a}}u_1,\\
\pi(u_2) &= \frac{1}{\sqrt{a}}u_2,\\
\pi(u_3) &= \frac{1}{\sqrt{a}}u_3,\\
\pi(u_4) &= \frac{1}{\sqrt{a}}u_4.
\end{aligned}$$

2°. $a = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_2$ we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

The proof of the Proposition is complete.

Proposition 4.3. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 4.3 is equivalent to one and only one of the following pairs:*

1.

$[,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	e_4
u_4	$-u_4$	0	u_3	$-u_1$	0	0	$-e_4$	0

2.

$[,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	0
u_4	$-u_4$	0	u_3	$-u_1$	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 4.3 is equivalent to the trivial.*

Proof. Let q be a virtual structure on generalized module 4.3. By Proposition 6, Chapter I, without loss of generality it can be assumed that q is primary.

Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_4, \quad \mathfrak{g}^{(2)}(\mathfrak{h}) = \mathbb{C}e_2, \quad \mathfrak{g}^{(-2)}(\mathfrak{h}) = \mathbb{C}e_3,$$
$$U^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_4, \quad U^{(-1)}(\mathfrak{h}) = \mathbb{C}u_2 \oplus \mathbb{C}u_3,$$

we have

$$C(e_1) = C(e_2) = C(e_3) = C(e_4) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Then

$$\begin{array}{llll}
[e_1, e_2] = 2e_2, & & & \\
[e_1, e_3] = -2e_3, & [e_2, e_3] = e_1, & & \\
[e_1, e_4] = 0, & [e_2, e_4] = 0, & [e_3, e_4] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = u_2, & [e_4, u_1] = 0, \\
[e_1, u_2] = -u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0, & [e_4, u_2] = 0, \\
[e_1, u_3] = -u_3, & [e_2, u_3] = -u_4, & [e_3, u_3] = 0, & [e_4, u_3] = -u_2, \\
[e_1, u_4] = u_4, & [e_2, u_4] = 0, & [e_3, u_4] = -u_3, & [e_4, u_4] = u_1.
\end{array}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I).

Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_4, \quad \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{C}e_2, \quad \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) = \mathbb{C}e_3,$$
$$\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{C}u_1 \oplus \mathbb{C}u_4, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{C}u_2 \oplus \mathbb{C}u_3.$$

Since

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}), \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}), \\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),
\end{aligned}$$

we have

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1 + a_2 e_4,\\
[u_1, u_3] &= b_1 e_1 + b_2 e_4,\\
[u_1, u_4] &= c e_2,\\
[u_2, u_3] &= d e_3,\\
[u_2, u_4] &= f_1 e_1 + f_2 e_4,\\
[u_3, u_4] &= k_1 e_1 + k_2 e_4.
\end{aligned}$$

Using the Jacobi identity we obtain:

$$a_1 = a_2 = b_1 = b_2 = c = d = f_1 = f_2 = k_1 = 0.$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3	u_4
e_1	0	$2e_2$	$-2e_3$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	$-2e_2$	0	e_1	0	0	u_1	$-u_4$	0
e_3	$2e_3$	$-e_1$	0	0	u_2	0	0	$-u_3$
e_4	0	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_4	0	u_2	0	0	0	k_2e_4
u_4	$-u_4$	0	u_3	$-u_1$	0	0	$-k_2e_4$	0 .

Consider the following cases:

1°. $k_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(e_3) &= e_3,\\
\pi(e_4) &= k_2 e_4,\\
\pi(u_1) &= k_2 u_1,\\
\pi(u_2) &= k_2 u_2,\\
\pi(u_3) &= u_3,\\
\pi(u_4) &= u_4.
\end{aligned}$$

2°. $k_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_2$ we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

The proof of the Proposition is complete.

Proposition 5.1. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 5.1 is equivalent to the trivial pair:*

1.

$[,]$	e_1	e_2	e_3	e_4	e_5	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	0	0	$-u_2$	u_1
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	0	0	0	0
u_3	u_3	u_3	u_4	0	u_2	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors e_1, e_2.

Lemma. *Any virtual structure q on generalized module 5.1 is equivalent to the trivial.*

Proof. Let q be a virtual structure on generalized module 5.1. By Proposition 6, Chapter I, without loss of generality it can be assumed that q is primary.

Since

$$\begin{aligned} &\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, && \mathfrak{g}^{(2,0)}(\mathfrak{h}) = \mathbb{C}e_3, \\ &\mathfrak{g}^{(-2,0)}(\mathfrak{h}) = \mathbb{C}e_4, && \mathfrak{g}^{(0,2)}(\mathfrak{h}) = \mathbb{C}e_5, \\ &U^{(1,1)}(\mathfrak{h}) = \mathbb{C}u_1, && U^{(-1,1)}(\mathfrak{h}) = \mathbb{C}u_2, \\ &U^{(-1,-1)}(\mathfrak{h}) = \mathbb{C}u_3, && U^{(1,-1)}(\mathfrak{h}) = \mathbb{C}u_4, \end{aligned}$$

we have

$$C(e_1) = C(e_2) = C(e_3) = C(e_4) = C(e_5) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 5.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Then

$$\begin{array}{lllll} [e_1, e_2] = 0, & & & & \\ [e_1, e_3] = 2e_3, & [e_2, e_3] = 0, & & & \\ [e_1, e_4] = -2e_4, & [e_2, e_4] = 0, & [e_3, e_4] = e_1, & & \\ [e_1, e_5] = 0, & [e_2, e_5] = 2e_5, & [e_3, e_5] = 0, & [e_4, e_5] = 0, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = u_1, & [e_3, u_1] = 0, & [e_4, u_1] = u_2, & [e_5, u_1] = 0, \\ [e_1, u_2] = -u_2, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, & [e_5, u_2] = 0, \\ [e_1, u_3] = -u_3, & [e_2, u_3] = -u_3, & [e_3, u_3] = -u_4, & [e_4, u_3] = 0, & [e_5, u_3] = -u_2, \\ [e_1, u_4] = u_4, & [e_2, u_4] = -u_4, & [e_3, u_4] = 0, & [e_4, u_4] = -u_3, & [e_5, u_4] = u_1. \end{array}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I).

Thus

$$\begin{aligned} &\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, && \bar{\mathfrak{g}}^{(2,0)}(\mathfrak{h}) = \mathbb{C}e_3, \\ &\bar{\mathfrak{g}}^{(-2,0)}(\mathfrak{h}) = \mathbb{C}e_4, && \bar{\mathfrak{g}}^{(0,2)}(\mathfrak{h}) = \mathbb{C}e_5, \\ &\bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}) = \mathbb{C}u_1, && \bar{\mathfrak{g}}^{(-1,1)}(\mathfrak{h}) = \mathbb{C}u_2, \\ &\bar{\mathfrak{g}}^{(-1,-1)}(\mathfrak{h}) = \mathbb{C}u_3, && \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{C}u_4. \end{aligned}$$

Since

$$
\begin{aligned}
[u_1, u_2] &\in \bar{\mathfrak{g}}^{(0,2)}(\mathfrak{h}),\\
[u_1, u_3] &\in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}),\\
[u_1, u_4] &\in \bar{\mathfrak{g}}^{(2,0)}(\mathfrak{h}),\\
[u_2, u_3] &\in \bar{\mathfrak{g}}^{(-2,0)}(\mathfrak{h}),\\
[u_2, u_4] &\in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}),\\
[u_3, u_4] &\in \bar{\mathfrak{g}}^{(0,-2)}(\mathfrak{h}),
\end{aligned}
$$

we have

$$
\begin{aligned}
[u_1, u_2] &= ae_5,\\
[u_1, u_3] &= be_1 + ce_2,\\
[u_1, u_4] &= de_3,\\
[u_2, u_3] &= fe_4,\\
[u_2, u_4] &= ke_1 + me_2,\\
[u_3, u_4] &= 0.
\end{aligned}
$$

Using the Jacobi identity we obtain: $a = b = c = d = f = k = m = 0$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

The proof of the Proposition is complete.

Proposition 6.1. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 6.1 is equivalent to one and only one of the following pairs:*

1.

$[,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	$2e_5$	$e_1 + e_2$	$2e_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	$-2e_5$	0	$2e_4$	$-e_1 + e_2$
u_3	u_3	u_3	u_4	0	u_2	0	$-e_1 - e_2$	$-2e_4$	0	$2e_6$
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	$-2e_3$	$e_1 - e_2$	$-2e_6$	0

2.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	0	0	0
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	0	0	0	0
u_3	u_3	u_3	u_4	0	u_2	0	0	0	0	0
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	0	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors e_1, e_2.

Lemma. *Any virtual structure q on generalized module 6.1 is equivalent to the trivial.*

Proof. Let q be a virtual structure on generalized module 6.1. By Proposition 6, Chapter I, without loss of generality it can be assumed that q is primary.

Since

$$\begin{gathered}
\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \\
\begin{aligned}
&\mathfrak{g}^{(2,0)}(\mathfrak{h}) = \mathbb{C}e_3, && \mathfrak{g}^{(-2,0)}(\mathfrak{h}) = \mathbb{C}e_4, \\
&\mathfrak{g}^{(0,2)}(\mathfrak{h}) = \mathbb{C}e_5, && \mathfrak{g}^{(0,-2)}(\mathfrak{h}) = \mathbb{C}e_6, \\
&U^{(1,1)}(\mathfrak{h}) = \mathbb{C}u_1, && U^{(-1,1)}(\mathfrak{h}) = \mathbb{C}u_2, \\
&U^{(-1,-1)}(\mathfrak{h}) = \mathbb{C}u_3, && U^{(1,-1)}(\mathfrak{h}) = \mathbb{C}u_4,
\end{aligned}
\end{gathered}$$

we have

$$C(e_1) = C(e_2) = C(e_3) = C(e_4) = C(e_5) = C(e_6) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 6.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Then

$$\begin{array}{llllll}
[e_1,e_2]=0, & & & & & \\
[e_1,e_3]=2e_3, & [e_2,e_3]=0, & & & & \\
[e_1,e_4]=-2e_4, & [e_2,e_4]=0, & [e_3,e_4]=e_1, & & & \\
[e_1,e_5]=0, & [e_2,e_5]=2e_5, & [e_3,e_5]=0, & [e_4,e_5]=0, & & \\
[e_1,e_6]=0, & [e_2,e_6]=-2e_6, & [e_3,e_6]=0, & [e_4,e_6]=0, & [e_5,e_6]=-e_2, & \\
[e_1,u_1]=u_1, & [e_2,u_1]=u_1, & [e_3,u_1]=0, & [e_4,u_1]=u_2, & [e_5,u_1]=0, & [e_6,u_1]=-u_4, \\
[e_1,u_2]=-u_2, & [e_2,u_2]=u_2, & [e_3,u_2]=u_1, & [e_4,u_2]=0, & [e_5,u_2]=0, & [e_6,u_2]=u_3, \\
[e_1,u_3]=-u_3, & [e_2,u_3]=-u_3, & [e_3,u_3]=-u_4, & [e_4,u_3]=0, & [e_5,u_3]=-u_2, & [e_6,u_3]=0, \\
[e_1,u_4]=u_4, & [e_2,u_4]=-u_4, & [e_3,u_4]=0, & [e_4,u_4]=-u_3, & [e_5,u_4]=u_1, & [e_6,u_4]=0.
\end{array}$$

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 7, Chapter I).

Thus

$$\begin{gathered}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2, \\
\begin{aligned}
&\bar{\mathfrak{g}}^{(2,0)}(\mathfrak{h}) = \mathbb{C}e_3, && \bar{\mathfrak{g}}^{(-2,0)}(\mathfrak{h}) = \mathbb{C}e_4, \\
&\bar{\mathfrak{g}}^{(0,2)}(\mathfrak{h}) = \mathbb{C}e_5, && \bar{\mathfrak{g}}^{(0,-2)}(\mathfrak{h}) = \mathbb{C}e_6, \\
&\bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}) = \mathbb{C}u_1, && \bar{\mathfrak{g}}^{(-1,1)}(\mathfrak{h}) = \mathbb{C}u_2, \\
&\bar{\mathfrak{g}}^{(-1,-1)}(\mathfrak{h}) = \mathbb{C}u_3, && \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{C}u_4.
\end{aligned}
\end{gathered}$$

Since

$$
\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(0,2)}(\mathfrak{h}),\\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}),\\
&[u_1, u_4] \in \bar{\mathfrak{g}}^{(2,0)}(\mathfrak{h}),\\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-2,0)}(\mathfrak{h}),\\
&[u_2, u_4] \in \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}),\\
&[u_3, u_4] \in \bar{\mathfrak{g}}^{(0,-2)}(\mathfrak{h}),
\end{aligned}
$$

we have

$$
\begin{aligned}
&[u_1, u_2] = ae_5,\\
&[u_1, u_3] = be_1 + ce_2,\\
&[u_1, u_4] = de_3,\\
&[u_2, u_3] = fe_4,\\
&[u_2, u_4] = ke_1 + me_2,\\
&[u_3, u_4] = ne_6.
\end{aligned}
$$

Using the Jacobi identity we obtain:

$$
\begin{cases}
a = d = f = n = 2c,\\
b = m = c,\\
k = -c.
\end{cases}
$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	0	$2e_3$	$-2e_4$	0	0	u_1	$-u_2$	$-u_3$	u_4
e_2	0	0	0	0	$2e_5$	$-2e_6$	u_1	u_2	$-u_3$	$-u_4$
e_3	$-2e_3$	0	0	e_1	0	0	0	u_1	$-u_4$	0
e_4	$2e_4$	0	$-e_1$	0	0	0	u_2	0	0	$-u_3$
e_5	0	$-2e_5$	0	0	0	$-e_2$	0	0	$-u_2$	u_1
e_6	0	$2e_6$	0	0	e_2	0	$-u_4$	u_3	0	0
u_1	$-u_1$	$-u_1$	0	$-u_2$	0	u_4	0	$2ce_5$	$ce_1 + ce_2$	$2ce_3$
u_2	u_2	$-u_2$	$-u_1$	0	0	$-u_3$	$-2ce_5$	0	$2ce_4$	$-ce_1 + ce_2$
u_3	u_3	u_3	u_4	0	u_2	0	$-ce_1 - ce_2$	$-2ce_4$	0	$2ce_6$
u_4	$-u_4$	u_4	0	u_3	$-u_1$	0	$-2ce_3$	$ce_1 - ce_2$	$-2ce_6$	0

Consider the following cases:

1°. $c \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= e_2,\\
\pi(e_3) &= e_3,\\
\pi(e_4) &= e_4,\\
\pi(e_5) &= e_5,\\
\pi(e_6) &= e_6,\\
\pi(u_1) &= \frac{1}{\sqrt{c}}u_1,\\
\pi(u_2) &= \frac{1}{\sqrt{c}}u_2,\\
\pi(u_3) &= \frac{1}{\sqrt{c}}u_3,\\
\pi(u_4) &= \frac{1}{\sqrt{c}}u_4.
\end{aligned}$$

2°. $c = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$.

Since $\dim \mathfrak{r}(\bar{\mathfrak{g}}_1) \neq \dim \mathfrak{r}(\bar{\mathfrak{g}}_2)$ we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

The proof of the Proposition is complete.

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